Fully non-linear equation:

$$F(x, y, z, p, q) = 0$$

Where as usual $z = u(x, y)$, $p = u'_x$ and $q = u'_y$, and

$$F'^2_p + F'^2_q \neq 0$$

The latter implies that locally either p or q can be found as a function of the remaining variables.

We have seen in the case F is linear with respect to p, q that the normal vector field to the graph $z = u(x, y)$ is orthogonal to the vector field $V = (-F'_p, -F'_q, 1)$. In that case the Cauchy problem $u|_\gamma = u_0(x, y)$ is well-posed if the curve γ is transversal to all characteristics it meets.

In order to adjust the characteristics method one needs to "linearize" the initial non-linear equation. An idea is to show that the first derivatives $p = u'_x$ and $q = u'_y$ satisfy *quasilinear* equations.

Namely, differentiating w.r.t. x and y yields two quasilinear equations for p, q:

$$F'_x + F'_z u'_x + F'_p p'_x + F'_q q'_x = 0 \qquad (1)$$

and

$$F'_y + F'_z u'_y + F'_p p'_y + F'_q q'_y = 0 \qquad (2)$$

Indeed, we have for the mixed partial derivatives: $p'_y = u''_{xy} = u''_{yx} = q'_x$, hence eq. (1) and (2) take the quasilinear form

$$F'_p p'_x + F'_q p'_y = -F'_x - F'_z p$$

$$F'_p q'_x + F'_q q'_y = -F'_y - F'_z q$$

Applying the characteristic equations to this system we get

$$\frac{dx}{dt} = F'_p, \qquad \frac{dy}{dt} = F'_q, \qquad \frac{dp}{dt} = -F'_x - F'_z p, \qquad \frac{dq}{dt} = -F'_y - F'_z q.$$

We need only one equation for z. One can get by differentiating $z = u(x, y)$ w.r.t. t subject to the first previous equations:

$$\frac{dz}{dt} = \frac{d}{dt} u(x, y) = u'_x \frac{dx}{dt} + u'_y \frac{dy}{dt} = p \cdot F'_p + q \cdot F'_q$$

Thus we have arrived at the following system

$$\frac{dx}{dt} = F'_p,$$

$$\frac{dy}{dt} = F'_q,$$

$$\frac{dz}{dt} = p \cdot F'_p + q \cdot F'_q$$

$$\frac{dp}{dt} = -F'_x - F'_z p,$$

$$\frac{dq}{dt} = -F'_y - F'_z q.$$

This system determines a family of integral curves in $\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$ and it is called the characteristic equations for the non-linear equation $F = 0$.

In **general**, in the n-dimensional case one has a system of similar equations in $\mathbb{R}^{2n+1}$. In fact, let we have a 1st order non-linear equation

$$F(x, u, Du) = 0$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $Du = \left(u_{x_1}, \ \dots, u_{x_n}\right) = (p_1, \dots, p_n)$ is the gradient of $u = u(x_1, \dots, x_n)$. Then the modified system for characteristics is

$$\frac{dx_k}{dt} = F'_{p_k},$$

$$\frac{dp_k}{dt} = -F'_{x_k} - F'_z p_k,$$

for $k = 1, \dots, n$, and

$$\frac{dz}{dt} = DF \ \cdot p = \sum_{k=1}^{n} F'_{p_k} p_k.$$

Return to $n = 2$. We must complete our Cauchy conditions because we have now 5 ODE's but only 3 initial Cauchy conditions. Since now we are in

$$\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$$

it is clear that we need only the Cauchy data p_0 and q_0.

- We recall that the Cauchy condition can be written as a parameterized curve:

$$\Gamma: \quad x = x_0(s),\ \ y = y_0(s),\ \ z = z_0(s)$$

  Substituting this into $F(x, y, z, p, q) = 0$ yields

$$F(x_0, y_0, z_0, p_0, q_0) = 0 \qquad \text{(IC-1)}$$

- Another relation is found by differentiating the original initial condition (IC) with respect to the inner parameter s:

$$\frac{d}{ds} z_0(s) = \frac{d}{ds} u\big(x_0(s), y_0(s)\big) = u'_x\big(x_0(s), y_0(s)\big) \cdot \frac{dx_0}{ds} + u'_y\big(x_0(s), y_0(s)\big) \cdot \frac{dy_0}{ds}$$

  This yields the so-called *strip condition*:

$$\frac{d}{ds} z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds} \qquad \text{(IC-2)}$$

These equations (IC-1) - (IC-2) provide two additional initial data, for p_0 and q_0.

In fact p_0 and q_0 need not to be uniquely defined and need not even exist. However, once p_0 and q_0 do exist, one can determine an integral surface

$$x = x(s, t),\ \ y = y(s, t),\ \ z = z(s, t)$$

which gives a parametric form for the solution of the Cauchy problem for the non-linear equation $F = 0$.

Remark: Our notation p_0 and q_0 here correspond to φ and ψ given in MacOwen, p. 34-35.

Method of envelopes

In general, for the 1st order non-linear equation

$$F(x, u, Du) = 0 \qquad (*)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, we set vector-notation

$$D_p F = (F_{p_1}, F_{p_2}, \dots, F_{p_n})$$

$$D_x F = (F_{x_1}, F_{x_2}, \dots, F_{x_n})$$

(We assume that F is smooth, at least of class C^2 in some domain in $\mathbb{R}^{2n+1}$).

We are concerned with finding solutions u of (*) in some open set $U \subset \mathbb{R}^n$, subject to the Cauchy condition

$$u = h \qquad \text{on } \Gamma,$$

where Γ is a subset of the boundary ∂U.

Suppose that we have found a parametric family of general solutions, say $u = u(x, a)$. Then we write also

$$(D_a u,\ D_{xa}^2) := \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{pmatrix}$$

for the composed Jacobian of size $n \times (n+1)$.

Definition: A function $u = u(x_1, \dots, x_n)$ of class C^2 is called a *complete integral* in $U \times A$ provided

(i) $u(x, a)$ solves (*) for each $a \in A$

and

(ii) rank $(D_a u,\ D_{xa}^2) = n$, $\qquad (x, a) \in U \times A$.

In other words, $u(x, a)$ depends on all the n independent parameters $a_1, \dots, a_n$.

Example 1. *Clairaut's equation* (in honor of Alexis C. de Clairault, 1713-1765)

$$x \cdot Du + f(Du) = u$$

For instance, if $n = 2$ one has

$$xu'_x + yu'_y + f\left(u'_x, u'_y\right) = u$$

Then a complete integral is

$$u(x, a) = a \cdot x + f(a)$$

Example 2. The eikonal equation from geometric optic is

$$|Du|^2 = u_{x_1}^2 + u_{x_2}^2 + \ ... + u_{x_n}^2 = 1$$

A complete integral is an *affine function*

$$u(x; a, b) = a \cdot x + f(b),$$

where $|a| = 1, \ \ b \in \mathbb{R}$.

Example 3. The *Hamilton-Jacobi equation* from mechanics (William R. Hamilton, 1805-1865 and Carl Jacobi, 1804-1851):

$$u'_t + H(Du) = 0, \qquad\qquad H: \ \mathbb{R}^n \to \ \mathbb{R}.$$

Here $u = u(x, t) = u(x_1, \dots, x_n, t)$, i.e. $t = x_{n+1}$. Then

$$u(x, t; a, b) = a \cdot x - tH(a) + b,$$

is a complete integral for $x \in \mathbb{R}^n$ and $t > 0$.

Theorem 1. *Let $u(x; a)$ be a complete integral for $F = 0$. Consider the vector equation*

$$D_a u(x; a) = 0 \qquad (**)$$

Suppose we can solve it for a as a smooth function of x: $a = \varphi(x)$. Then the ***envelope function*** *$v(x) = u(x;\ \varphi(x))$ solves also the original equation $F = 0$.*

Remark: The method also works if one replaces one parameter, say a_n by a function of the remaining parameters, and substitute it into $u(x; a)$. This yields in general a wide choice of envelope solutions.

Idea of the proof: We have

$$v'_{x_k}(x) = \frac{\partial}{\partial x_k} u\big(x;\ \varphi(x)\big) = u'_{x_k}\big(x;\ \varphi(x)\big) + \sum_{i=1}^{n} u'_{a_i}\big(x;\ \varphi(x)\big) \cdot \frac{\partial \varphi_i}{\partial x_k}$$

where $u'_{a_i} = 0$ for $a = \varphi(x)$ by virtue of our assumption (**). Hence

$$v'_{x_k}(x) = \frac{\partial}{\partial x_k} u\big(x;\ \varphi(x)\big), \qquad k = 1, \dots, n$$

and it easily follows that the envelope function satisfies also $F(x, v, Dv) = 0$. ■

How it works?

We return again to $n = 2$. Then a complete integral is denoted by $u(x, y; a, b)$ and it depends on independent parameters a and b. The above rank-condition is equivalent to saying that mapping

$$(a, b) \longrightarrow (u, u'_x, u'_y)$$

has rank 2 at each fixed x and y, that is the matrix

$$\begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix}$$

has maximal rank.

In practice one usually uses a one parametric envelope solution which can be found by substituting some auxiliary function $b = B(a)$ or $a = A(b)$ in $u(x, y; a, b)$. We demonstrate this below.

Example 5. Consider

$$u'_x = {u'_y}^2$$

subject to initial condition $u(0, y) = \frac{y^2}{2}$.

Solution by the envelope method.

(i) An idea is to find solutions in the class of linear forms of the kind:

$$v = a + bx + cy + dxy.$$

The straightforward computation yields $d = 0$, $b = c^2$, while a can be chosen arbitrarily. This gives after changing notation

$$v = a + b^2x + by$$

We see that the our Jacobian matrix has rank 2 (the first two columns):

$$\begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2bx + y & 2b & 1 \end{pmatrix}$$

Hence now we are in position of Theorem 1.

(ii) Set $a = kb^2$, where the constant k will be chosen later. We have

$$v = kb^2 + b^2x + by$$

and the envelope equation is

$$0 = \frac{\partial}{\partial b} v = 2kb + 2bx + y,$$

hence $b = -\frac{y}{2x+2k}$

(iii) Substituting this into v we find

$$v(x, y; a, b) = -\frac{y^2}{4(x + k)}$$

(iv) Finally applying our Cauchy condition we find $k = -\frac{1}{2}$. Hence the desired solution is

$$u(x, y) = \frac{y^2}{2 - 4x}$$

Question: Why $a = kb^2$? Check that the above argument breaks down for $a = kb$

Example 5. Consider

$$u'_x u'_y = u$$

Analys:

- $u = xy + ax + by + ab$ is a complete integral
- $u'_a = x + b, u'_b = y + a,$ hence we find $a = -y$ and $b = -x.$ This is the function φ in the Theorem.
- substituting φ into u yields: $u = xy + ax + by + ab = 0$
  which provides us another, trivial, solution.

Another choice is $b = a$. Then we get

$$u = xy + ax + ay + a^2$$

and $0 = u'_a = x + y + 2a$, hence $a = -\frac{x+y}{2}$.

Substituting this into u yields $u = -\frac{(x-y)^2}{4}$.