Lecture 4: Introduction in higher order PDE

Multi-index notation

- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)\,,\ \ \alpha_k \in \mathbb{N}$
- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$
- $\alpha! = \alpha_1!\,\alpha_2! \dots \alpha_n!$

Products and derivative:

- $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
- $x^\alpha = x^{\alpha_1} x^{\alpha_2} \dots x^n\,,$
- $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$

Commutative relations:

$$D^\alpha D^\beta = D^\beta D^\alpha = D^{\alpha+\beta}$$

Definition: A general m-order PDE is an equation of the kind

$$F(x, D^\alpha u) = 0\,,\ \text{where } |\alpha| \le m \qquad (*)$$

Cauchy problem:

- For the 1st order PDE with 2 variables we consider Cauchy $u|_\gamma$ data on a curve γ in $\mathbb{R}^2$ or on a surface for 3 variables, etc.
- It is natural to assume that Cauchy problem for (*) involves replacing the initial curve γ in $\mathbb{R}^2$ by initial surface S in $\mathbb{R}^n$, but we still need to involve some additional data, say the normal derivatives of u along S.
- What does it mean that S is noncharacteristic in this new set-up?

Definition: By the Cauchy data for the m-th order PDE (*) we understand the set of values

$$u,\ \ \frac{\partial u}{\partial \nu},\ \ \frac{\partial^2 u}{\partial \nu^2}, \dots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \qquad (**)$$

along a *hypersurface* S, where ν is the unit *normal vector* to S. Moreover, the surface S is called *noncharacteristic* for (*) if the derivatives (**) on S determine all derivatives of the solution u on S.

This noncharacteristic property will be true, if for example, we can express the initial equation in the form

$$\frac{\partial^m u}{\partial \nu^m} = G(x, D^\alpha u)$$

where the R.H.S. does not contain $\frac{\partial^m u}{\partial \nu^m}$.

The latter form is called the *normal form* of (*) with respect to the hypersurface *S.*

By local changing of the coordinates $(x_1, x_2, \dots, x_n)$, one can assume that S is "straightened out", i.e. it coincides with a hyperplane in new coordinates.

Example: Consider a surface S given by $x_3 = x_1 x_2 + 1$. Then in the new coordinates

$$\tilde{x}_1 = x_1, \quad \tilde{x}_2 = x_2, \quad \tilde{x}_3 = x_3 - x_1 x_2 - 1$$

the surface is given as $\tilde{x}_3 = 0$. It is easy to show that the Jacobian $\frac{\partial(\tilde{x}_1, \tilde{x}_1, \tilde{x}_1)}{\partial(x_1, x_2, x_3)} = 1 \neq 0$.

Thus, without loss of generality we can assume that S is given by $x_n = 0$ in some coordinates $(x_1, \dots, x_n)$ and therefore the normal derivatives coincide with the corresponding derivative w.r.t. the distinguished coordinate x_n:

$$\frac{\partial^k u}{\partial \nu^k} = \frac{\partial^k u}{\partial x_n^k}$$

Hence our Cauchy problem is rewritten in this "flat" case as follows:

$$\left.\frac{\partial^k u}{\partial x_n^k}\right|_{x_n=0} = g_k(x_1, \dots, x_{m-1}), \qquad \text{for all } k = 0, 1, 2, \dots, m-1.$$

- It turns out that for en equation given in the normal form, the Cauchy data determines *all* derivatives of the solution u on S. Indeed, the derivatives are found step by step, by differentiating the initial conditions and the normal equation.

A deeper result is the celebrated

Cauchy-Kovalevski Theorem: *If all* $g_k,\ k = 0,1, \dots, m-1$ *are real* analytic *in a neighborhood of* $0 \in \mathbb{R}^{n-1}$ *, and if G is real analytic in a neighborhood of* $(0, D^\alpha u(0))$*, then there exists a unique real analytic solution u of the above Cauchy problem in some neighborhood of* $0 \in \mathbb{R}^n$*.*

Shortcomings of Cauchy-Kovalevski Theorem:

- The theorem is non-effective in practical questions
- It fails to recognize well posed non-analytic Cauchy problems

What is a well posed problem?

In physics, one expects a stability of solutions with respect to their initial conditions, because a small change of data should induce only a small change in the solution. Otherwise, the solution becomes meaningless.

Definition: A problem is well posed (in the sense of Hadamard) if a solution exists, is unique, and depends continuously on its data.

Lewy example: there exists a *smooth* complex-valued function $F(x,y)$ such that the differential equation

$$u'_x + ixu'_y = F(x,y)$$

has no solutions. Hence, the analog of the Cauchy-Kovalevskaya theorem fails for equations with the smooth (infinitely many times differentiable) coefficients.

Classification of the second-order equations in two variables

Consider a *principally linear* equation, that is

$$a(x,y)u''_{xx} + \ b(x,y)u''_{xy} + c(x,y)u''_{yy} = d(x,y,u,u'_x,u'_y) \qquad \text{(PDE)}$$

Then the L.H.S. is called the *principal part* of (PDE), or its principal symbol.

We find conditions for $\gamma \subset \mathbb{R}^2_{xy}$ to be a characteristic curve. Consider a Cauchy data along γ:

$$u|_\gamma = h, \ \ \frac{\partial u}{\partial \nu}|_\gamma = h_1$$

where ν is the unit *normal vector* to γ. Equivalently one can write

$$u|_\gamma = h, \quad \frac{\partial u}{\partial x}|_\gamma = \varphi, \quad \frac{\partial u}{\partial y}|_\gamma = \psi$$

subject to the following *additional* compatibility condition

$$h'(s) = \varphi(s)f'(s) + \psi(s)g'(s)$$

where $(f(s), g(s))$ is a parameterization of γ.

We have

$$\varphi' = (u'_x)'_s = u''_{xx}f' + u''_{xy}g'$$

$$\psi' = \left(u'_y\right)'_s = u''_{xy}f' + u''_{yy}g'$$

Hence, after combining these equations with (PDE), we obtain a linear system on the second derivatives which is uniquely solvable provided that

$$D = \begin{vmatrix} f' & g' & 0 \\ 0 & f' & g' \\ a & b & c \end{vmatrix} = ag'^2 - bf'g' + c\,f'^2 \neq 0$$

In particular, the Cauchy data determines all second-order derivatives along γ iff $D \neq 0$.

Define the following quadratic form:

$$\sigma(\xi_1, \xi_2; x, y) = a(x,y)\xi_1^2 + b(x,y)\xi_1\xi_2 + c(x,y)\xi_2^2$$

One distinguishes the following three cases (depending on the point (x, y):

(i) $b^2 - 4ac > 0$, there are two characteristics, and (PDE) is called *hyperbolic*
(ii) $b^2 - 4ac = 0$, there are only one characteristic, and (PDE) is called *parabolic*
(iii) $b^2 - 4ac < 0$, there are no characteristics, and (PDE) is called *elliptic*

Examples:

- The wave equation $u''_{xx} - u''_{yy} = 0$ is *a hyperbolic equation*
- The heat equation $u''_{xx} - u'_y = 0$ is *a parabolic equation*
- The Laplace equation $u''_{xx} + u''_{yy} = 0$ is *an elliptic equation*

How to find characteristics?

If γ is given as a graph $y = y(x)$ then it is defined by an ordinary differential equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

How to reduce a principal linear PDE to a canonic form?

Usually, one solves the above ODE to determine exact solutions $y_1(x)$ and $y_2(x)$. Then $\lambda(x, y) = y_1(x) - y$ and $\mu(x, y) = y_2(x) - y$ are new coordinates which "diagonalize" the principal part. After that one translates the old derivatives u'_x, u''_{xx} etc. into the derivative w.r.t. λ and μ.

Obs.!: Instead of the given above $\lambda(x, y)$ and $\mu(x, y)$ one can try a more suitable pair $\tilde{\lambda} = \varphi(\lambda)$ and $\tilde{\mu} = \psi(\lambda)$, where $\varphi', \psi' \neq 0$.

Example (tentamen, 1998-03-16)

Study the equation

$$\frac{3}{4}u''_{xx} - 2y\, u''_{xy} + y^2 u''_{yy} + \frac{1}{2}u'_x = 0$$

a) *Where the equation is hyperbolic?*
b) *Determine the characteristic curves.*
c) *Transform the equation to canonical form where this is possible.*
d) *Determine the general solution in the domain where it is hyperbolic.*

Solution.

a) The principal symbol is $\frac{3}{4}\xi_1^2 - 2y\,\xi_1\xi_2 + y^2\xi_2^2$ with discriminant

$$b^2 - 4ac = 4y^2 - 3y^2 = y^2 \geq 0.$$

Hence our equation is *hyperbolic* everywhere outside the x-axis, that is when $y \neq 0$, where our equation has parabolic type. Denote by $U = \{(x, y): y \neq 0\}$.

b) The characteristic equation has the form:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2y \pm y}{3/2} = \{-2y, -\frac{2}{3}y\}$$

We assume that y >0 (the remained case is symmetric). Then by solving $y'(x) = -2y$ and $y'(x) = -\frac{2}{3}y$ we find equations for the characteristic curves:

$$\lambda := \ln y + 2x = const, \qquad \mu := 3\ln y + 2x = const$$

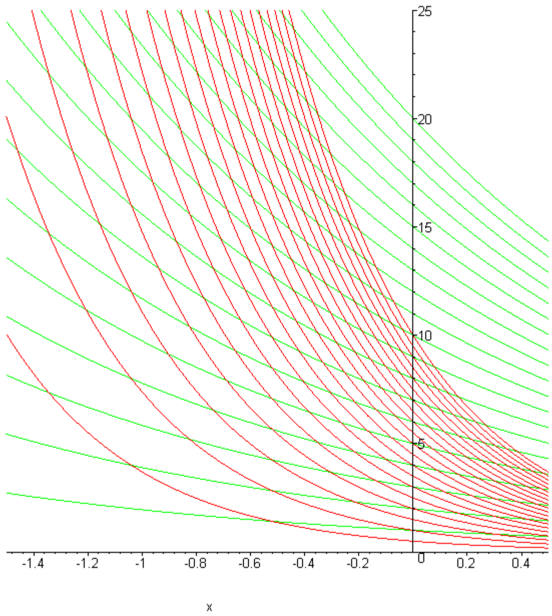

Figure 1 The characteristic lines: $\boldsymbol{\mu = const}$ colored by green and $\boldsymbol{\lambda = const}$ colored by green

c) We have $y = E^{-1}$, where $E = \exp\frac{\lambda-\mu}{2}$ and $x = \frac{3\lambda-\mu}{4}$. In the new coordinates

$$u'_x = 2u'_\lambda + 2u'_\mu$$

$$u'_y = \frac{1}{y}\cdot u'_\lambda + \frac{3}{y}\cdot u'_\mu = (u'_\lambda + 3\,u'_\mu)\cdot\exp\frac{\lambda-\mu}{2}$$

and setting , we find

$$u''_{xx} = 2\cdot(u'_x)'_\lambda + 2\cdot(u'_x)'_\mu = 4u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 4u''_{\mu\mu}$$

$$u''_{xy} = 2\cdot[(u'_\lambda + 3\,u'_\mu)E]'_\lambda + 2\cdot[(u'_\lambda + 3\,u'_\mu)E]'_\mu = (2u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 6u''_{\mu\mu} + 2u'_\mu)E$$

$$u''_{yy} = [(u'_\lambda + 3\,u'_\mu)E]'_\lambda \cdot E + 3\cdot[(u'_\lambda + 3\,u'_\mu)E]'_\mu\cdot E = (4u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 4u''_{\mu\mu})E^2$$

Hence substituting the found relations into initial equation yields

$$\frac{3}{4}u''_{xx} - 2y\,u''_{xy} + y^2u''_{yy} + \frac{1}{2}u'_x = -4u''_{\lambda\mu} - 4u'_\mu = 0$$

We have

$$u''_{\lambda\mu} + u'_\mu = 0$$

is the desired canonical form.

d) In order to find the general solution we write the latter equation as $(u'_\lambda + u)'_\mu = 0$, hence

$$u'_\lambda + u = f(\lambda)$$

for arbitrary $f(\lambda)$. Solving this linear ODE yields

$$u = g(\mu)e^{-\lambda} + f_1(\lambda)$$

for a new function f_1. Thus the general solution is found as

$$u = F_1(\mu)e^{-\lambda} + F_2(\lambda)$$

where both function can be chosen arbitrarily. ■