Lecture 5: The wave equation, dim=1

The one-dimensional wave equation is a hyperbolic 2nd order PDE of the form

$$u''_{tt} - c^2 u''_{xx} = 0$$

It describes the propagation of waves with a constant speed $c \neq 0,\ c \in \mathbb{R}$.

The characteristics for the wave equation are $x \pm ct = \text{const}$, and the change of variables

$$\lambda := x + ct, \qquad \mu := x - ct$$

transforms the initial equation to its canonical form

$$u_{\lambda\mu} = 0.$$

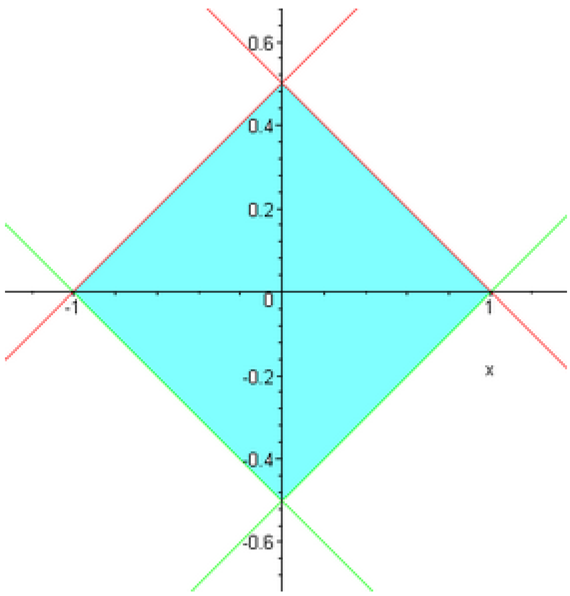
Then the *general solution* of the latter equation is

$$u(x,t) = F(x + ct) + G(x - ct). \qquad (*)$$

If we assume that $F(s)$ and $G(s)$ are of class C^2 in the interval $(a, b) \subset \mathbb{R}$ then $u(x,t)$ is of class C^2 in a rectangle domain

$$a < x \pm ct < b,$$

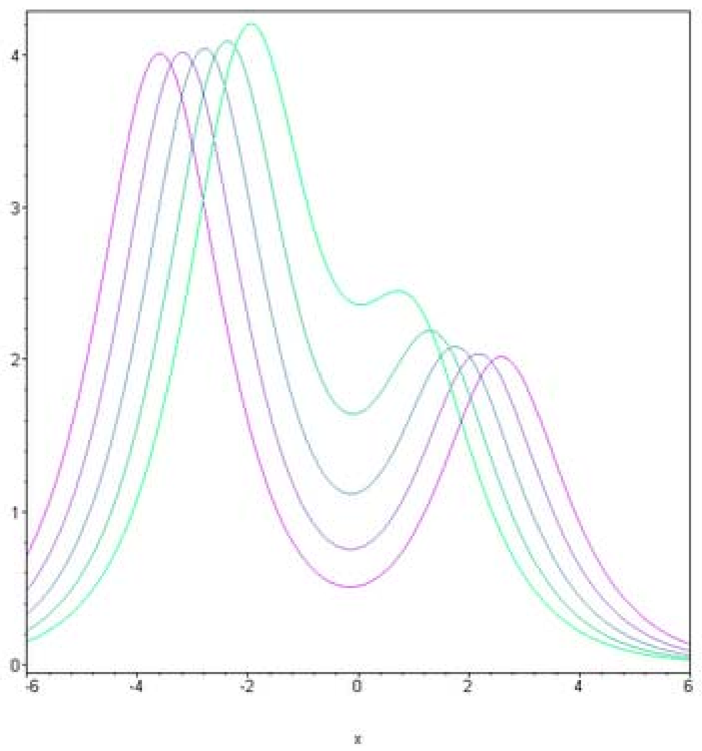
(see the picture below) which is called rectangles of characteristics:



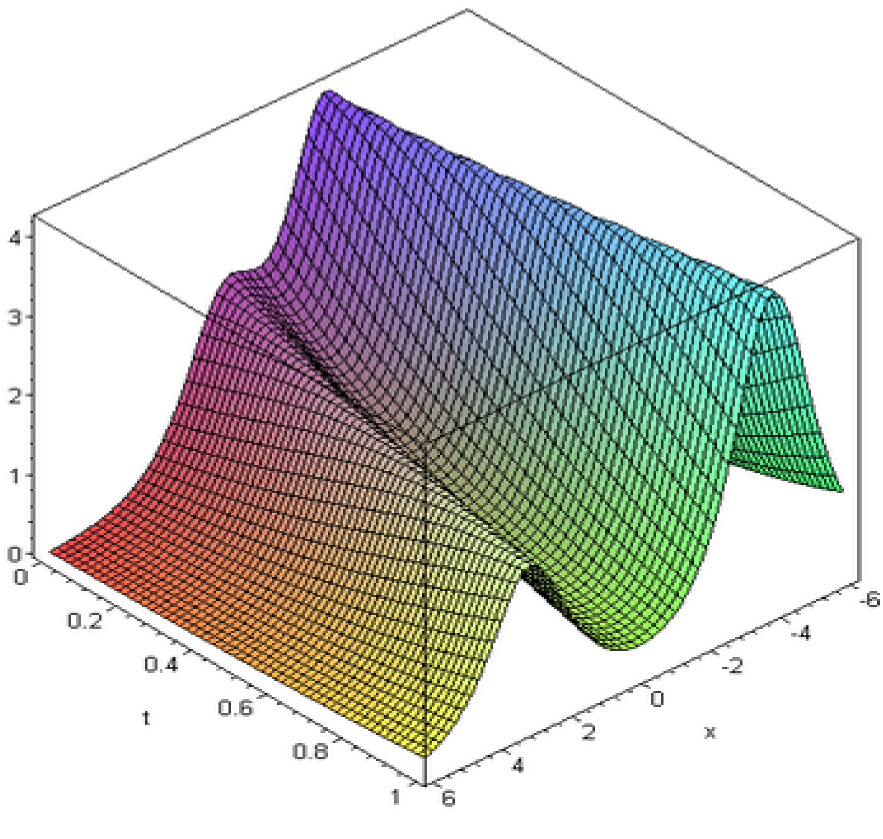
The fact that $u(x,t)$ is a sum of two functions in one variable usually is interpreted as a superposition of two waves propagating with a constant shape in *opposite* directions along the x-axis.

Example: In the pictures below you can see the superposition of waves with

$$f(x) = \frac{1}{\cosh(x-1)}, \qquad g(x) = \frac{4}{\cosh(x+2)}$$



and the corresponding time evolution:



The initial value problem

Let us consider the following Cauchy problem for the above wave equation:

$$u(x,0) = g(x), \qquad u_t'(x,0) = h(x),$$

where g and h are arbitrary functions. Using the representation (*), we get

$$u(x,0) = F(x) + G(x)$$

and

$$u_t'(x,0) = cF'(x) - cG'(x).$$

By integrating the later,

$$c[F(x) - G(x)] = \int_0^x h(s)\,ds + C$$

and combining this with the former equation we find from the obtained linear system that

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x h(s)\,ds + C_1$$

$$G(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x h(s)\,ds - C_1$$

for some new constant C_1. Substituting the found relations yields the well-known

d' Alembert's formula:

$$u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$

In the other direction, a straightforward calculation shows that the above representation gives a C^2-solution to the above Cauchy problem provided that $g \in C^2$ and $h \in C^1$.

Some conclusions from d'Alembert's formula

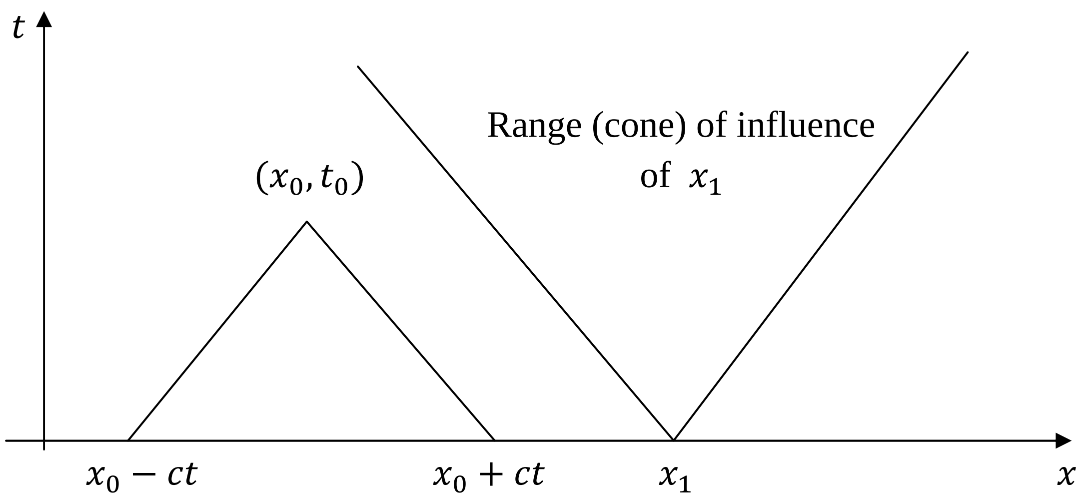
- The smoothness of $u(x,t)$ is prescribed by that of its initial conditions, for instance, if $g \in C^{p+1}$ and $h \in C^p$ then $u(x,t)$ is a C^{p+1}-solution.

- The solution is unique and $u(x,t)$ depends continuously on the data (obs. that there are not derivatives in the R.H.S.!). Hence the Cauchy problem for the wave equation is *well posed*.

- One can see from the d'Alembert formula (see also the picture above) that the solution at some point (x_0, t_0), where $x_0 \in \mathbb{R}$, $t_0 \geq 0$, is completely determined by the initial data in the following interval (the *domain of dependence* for (x_0, t_0)):

$$x_0 - ct_0 \leq x \leq x_0 + ct_0$$

Physically, this property is equivalent to the finite propagation speed of signals:



- The later property provides also the background of *special relativity* (considered firstly by Hendrik Lorentz and Henri Poincaré, and later by Albert Einstein). The cone above then is the so-called light cone in the space-time.

Weak solutions

In view of the general solution given above,

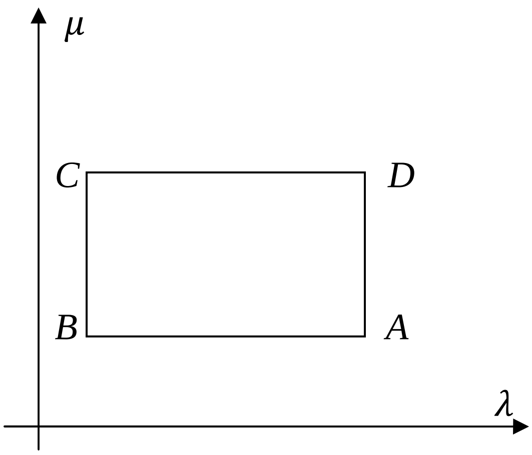
$$u(x,t) = F(x+ct) + G(x-ct)$$

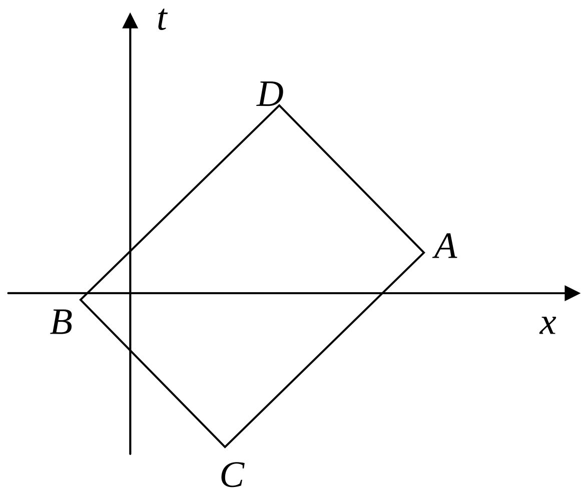
it is natural to expect that it defines a *weak* solution when the functions F and G are no longer of class C^2. There are several ways to define a weak solution to the wave equation, we consider an algebraic approach which requires no extra analytic considerations.

Recall our notation

$$\lambda = x + ct, \qquad \mu = x - ct,$$

and consider some functions $F(\lambda)$ and $G(\mu)$ and a rectangle $ABCD$ in the $\lambda\mu$-plane as shown in the **first** picture below:





Since $F(\lambda)$ is constant along vertical lines and $G(\mu)$ is constant along horizontal lines, we have

$$F(A) = F(D), \quad F(B) = F(C), \quad G(A) = G(B), \quad G(C) = G(D).$$

Using our representation $u(\lambda,\mu) = F(\lambda) + G(\mu)$ we find

$$u(A) + u(C) = u(B) + u(D) \qquad (**)$$

that is the sums of the values of u at opposite vertices are equal. Translated to the xt-plane, we view the previous relation as a parallelogram rule for solutions (recall that the sides of the latter parallelogram are segments of characteristics).

Definition: a weak solution of the wave equation is any function $u(x,t)$ satisfying (**) for every such parallelogram in its domain.