Lecture 6: The wave equation (cont.)

Finite string with fixed ends: initial/boundary value problems

The simplest interpretation of the 1-dim wave equation is the string model of finite length (the so-called vibrating string). In this model, $u(x,t)$ measures the distance from the equilibrium of the mass situated at point x and at the time t. After a suitable idealization, combination of Hooke's and Newton's laws readily yield the wave equation for $u(x,t)$.

In this model, it is natural then to consider a vibrating string on the x-interval $[0, L]$, with Cauchy data for u at $t = 0$ (the shape of a string and the initial velocity of the string), and some boundary conditions for u at the "strings ends" for $x = 0$ and $x = L$.

For example, the string with "fixed ends" is described by the following conditions:

$$u''_{tt} - c^2 u''_{xx} = 0$$

$$u(x,0) = g(x), \quad u'_t(x,0) = h(x), \qquad \text{for} \qquad 0 < x < L,$$

$$u(0,t) = u(L,t) = 0, \qquad \text{for } \ t \geq 0.$$

Fourier method

One approach to solve the above problem is to expand $u(x,t)$ in Fourier series with coefficients depending on time t:

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin \frac{k\pi\, x}{L}$$

(This is possible if $a_k(t)$ are "small enough"; for example, if the coefficients can be majorized by some rapidly decreasing number sequence). Then one finds that

$$a''_k + \left(\frac{k\pi\, c}{L}\right)^2 a_k = 0$$

The general solution of the latter equation is

$$a_k(t) = c_k \sin \frac{k\pi\, ct}{L} + d_k \cos \frac{k\pi\, ct}{L},$$

so we can determine coefficients c_k and d_k by the initial conditions. Namely,

$$g(x) = \sum_{k=1}^{\infty} d_k \sin\frac{k\pi x}{L}, \qquad h(x) = \sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{L},$$

with standard Fourier formulas, one finally finds that

$$d_k = \frac{2}{L}\int_0^L g(x) \sin\frac{k\pi x}{L}\, dx, \qquad c_k = \frac{2}{k\pi c}\int_0^L h(x) \sin\frac{k\pi x}{L}\, dx.$$

A reflection method

Another approach is the so-called ***Reflection method*** (see McOwen, p. 78-79). We explain it for the case when the initial/boundary problem for the wave equation

$$u''_{tt} - u''_{xx} = 0$$

is given in the wedge $x \geq 0,\ t \geq 0$:

$$u(x,0) = g(x), \quad u'_t(x,0) = h(x), \qquad x > 0,\ t = 0,$$

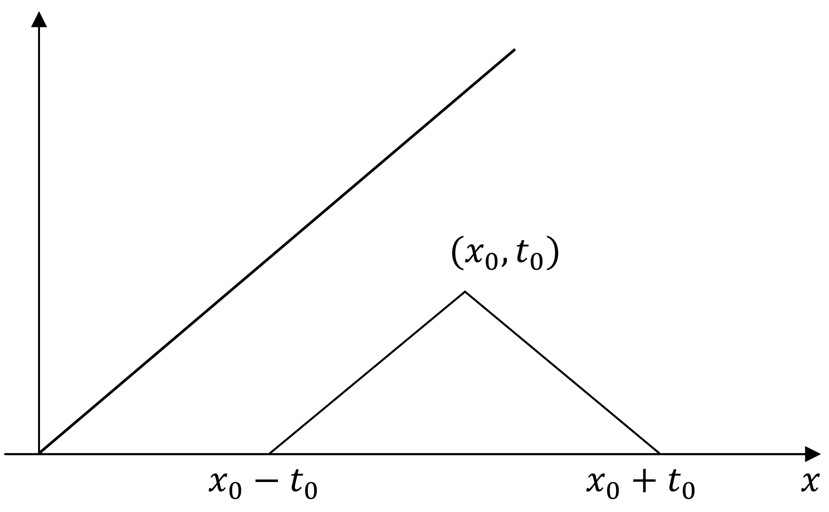
$$u(0,t) = 0, \qquad x = 0, \quad t > 0$$

and the compatibility condition holds:

$$g(0) = h(0) = 0.$$

Solution:

Method I. For any point (x_0, t_0) in the infinite triangle $x \geq\ t \geq 0$ (see the picture below) $u(x_0, t_0)$ can be found by the d'Alembert formula by using the initial problem on interval $(x_0 - t_0, x_0 + t_0)$ (observe that the left end of the interval is positive by our choice of $(x_0, t_0\,)$)



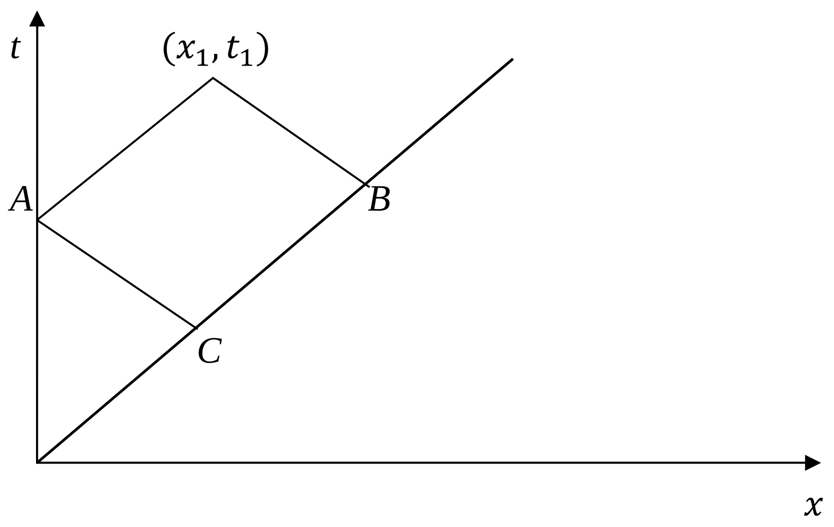
We have:

$$u(x_0, t_0) = \frac{1}{2}[g(x_0 + t_0) + g(x_0 - t_0)] + \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} h(s)\, ds.$$

For the further convenience we notice that along the characteristic $x = t$ we have

$$u(a, a) = \frac{g(2a)}{2} + \frac{1}{2}\int_0^{2a} h(s)\, ds. \qquad (**)$$

In the upper triangle, $0 \leq x \leq t$ we also consider an arbitrary point (x_1, t_1) and draw characteristics from this point, and reflect one characteristic which meets the t-axis as shown in the picture:



Then by parallelogram rule we have

$$u(x_1, t_1) = u(A) + u(B) - u(C)$$

Here $A = (0, t_1 - x_1)$, $B = (\frac{x_1+t_1}{2}, \frac{x_1+t_1}{2})$, $C = (\frac{t_1-x_1}{2}, \frac{t_1-x_1}{2})$.

Hence applying

$$u(x_1, t_1) = u(0, t_1 - x_1) + u\left(\frac{x_1+t_1}{2}, \frac{x_1+t_1}{2}\right) - u(\frac{t_1-x_1}{2}, \frac{t_1-x_1}{2})$$

An taking into account the boundary condition $u(0, t_1 - x_1) = 0$ and (**), we obtain

$$u(x_1, t_1) = \frac{1}{2}[g(x_1 + t_1) - g(x_1 - t_1)] + \frac{1}{2c}\int_{x_1-t_1}^{x_1+t_1} h(s)\, ds,$$

Method II. A key idea is to use the symmetry in the above conditions to extend all the functions by *odd reflection*:

$$\tilde{u}(x,0) = -u(-x,t) \qquad \text{for } \; x \le 0, \; t \ge 0$$

$$\tilde{u}(x,0) = u(x,t) \qquad \text{for } \; x > 0, \; t \ge 0$$

and define $g(x) = -g(-x)$ for $x \le 0$ and $\tilde{g}(x) = g(x)$ for $x > 0$. Similar we extend $h(s)$ to $\tilde{h}(s)$. It is then easy to see that these extensions are consistent with the given conditions. In this new setting, we have the standard Cauchy problem

$$\tilde{u}(x,0) = g(x), \qquad \tilde{u}_t'(x,0) = h(x),$$

and d'Alembert formula implies

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+ct) + \tilde{g}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\, ds$$

Returning to the old notation we obtain

$$u(x,t) = \begin{cases} \dfrac{1}{2}[g(x+ct) + g(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} h(s)\, ds, & \text{if } \; x \ge t \ge 0 \\ \dfrac{1}{2}[g(x+ct) - g(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} h(s)\, ds, & \text{if } \; 0 \le x \le t \end{cases}$$

The nonhomogeneous wave equation: the Duamel[1] Principle

Now consider the nonhomogeneous wave equation

$$u''_{tt} - c^2 u''_{xx} = f(x,t)$$

First we suppose we want to solve this equation subject to the homogeneous initial conditions:

$$u(x,0) = u'_t(x,0) = 0.$$

Remark: It is easy to see that any solution to the nonhomogeneous equation can be found as a sum of a solution with homogeneous initial conditions above and a solution to homogeneous equation

$$u''_{tt} - c^2 u''_{xx} = 0$$

with the given initial conditions.

Consider the following auxiliary problem:

$$U''_{tt} - c^2 U''_{xx} = 0, \qquad \text{for} \quad x \in \mathbb{R}, \quad t > 0$$

$$U(x,0,s) = 0, \qquad \text{for} \quad x \in \mathbb{R}$$

$$U'_t(x,0,s) = f(x,s), \qquad \text{for} \quad x \in \mathbb{R}$$

where $s > 0$.

Duhamel's Principle. *If* $U(x,t,s)$ *is a* $C^2(x \in \mathbb{R}, \quad t > 0)$ *and continuous in* s, $s > 0$, *and it solves the above auxiliary problem, then*

$$u(x,t) = \int_0^t U(x, t-s, s) ds$$

solves the homogeneous initial problem for the nonhomogeneous wave equation.

Corollary. *If the R.H.S. of the nonhomogeneous wave equation* $f(x,t)$ *is* C^1 *in* x *and* C^0 *in* t *then an explicit form of the solution is given by*

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(\xi, s)\, d\xi\, ds$$

---

[1] Jean-Maria-Constant Duhamel (1797–1872), a French mathematician

Higher dimensions: spherical means

Let $h(x)$ be continuous on $\mathbb{R}^n$, $n \geq 2$. The its spherical mean or average on a sphere of radius r centered at x is

$$M_h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r\xi)\, dS_\xi$$

- ω_n denotes the area of the unit sphere in $\mathbb{R}^n$,

$$\omega_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{\mathrm{n}}{2}\right)}$$

- dS_ξ denotes the surface measure.
- Since h is continuous in x,

$$\lim_{r \to 0+} M_h(x,r) = h(x)$$

The main property of $M_h(x,r)$ is the following identity, called also the *Darboux equation*:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_h(x,r) = \Delta_x M_h(x,r)$$

The latter identity allows to reduce the Cauchy problem for the *n*-dim wave equation to a partial differential equation in two variables.

For *n*=3 we have the *Kirchhoff formula*

$$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|\xi|=1}^{t} g(x + ct\,\xi) dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1}^{t} h(x + ct\,\xi) dS_\xi$$

which solves

$$u''_{tt} = c^2 \Delta u, \qquad \text{for} \quad x \in \mathbb{R}^n,\ t > 0$$

with initial conditions

$$u(x,0) = g(x), \quad u'_t(x,0) = h(x), \quad \text{for} \quad x \in \mathbb{R}^n.$$