Lecture 7: The wave equation: higher dimensional case

Some auxiliary facts

The co-area formula. Let $D \subset \mathbb{R}^n$ be a bounded domain and $f(x)$ be a function of class $C^1(D)$. Denote by $D_f(\xi) = \{x \in D\colon\ f(x) = \xi\}$ the ξ-level set of f. Let $h(x) \in L^1(D)$ be any integrable function. Then for almost every t the set $D_f(\xi)$ is a regular hypersurface and

$$\int_D h(x)dx = \int_{-\infty}^{+\infty} d\xi \int_{D_f(\xi)} \frac{h(y)}{|\nabla f(y)|} dS_y,$$

where dS_y is the surface measure on $D_f(\xi)$. In fact, the exterior integral should be taken only over the interval of those t for which $D_f(\xi)$ is non-empty.

Example 1. When $f(x) = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the distance function we have

$$\nabla f(x) = \nabla |x| = \frac{x}{|x|} \qquad \Longrightarrow \qquad |\nabla f(x)| = 1$$

Denote by

$$D = B_x(r) = \{y \in \mathbb{R}^n\colon\ |y - x| < r\}$$

the n-dimensional ball centered at x of radius r. In this case $D_f(\xi)$ is non-empty only for $0 \le \xi < r$ and the level sets are spheres $\partial B_x(\xi)$. Then applying the co-area formula gives

$$\int_{B_x(r)} h(x)dx = \int_0^r d\xi \int_{\partial B_x(\xi)} h(x)\, dS_y.$$

This formula is well-known (the integral in polar coordinates).

The divergence theorem. Let $D \subset \mathbb{R}^n$ be an open subset with a piecewise smooth boundary and let F be a continuously differentiable vector-field defined in a neighborhood of D. Then

$$\int_D \operatorname{div} F\, dx = \int_{\partial D} (F \cdot \nu)\, dx$$

Where ν is the unit outward normal vector-field along the boundary ∂D.

Spherical means

We return to the wave equation in $\mathbb{R}^n$ and consider the following initial-value problem

$$u''_{tt} - \Delta_x u = 0$$

$$u(x,0) = g(x), \quad u'_t(x,0) = h(x), \qquad x \in \mathbb{R}^n, \; t = 0$$

Define the spherical means, or averages, of a function $u(x,t)$ by

$$U(x,r,t) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_x(r)} u(y,t)\, dS_y$$

where $\omega_n r^{n-1}$ is the $(n-1)$-dimensional area of the $(n-1)$-dimensional sphere of radius r. Here ω_n is the $(n-1)$-dimensional area of the unit sphere $\partial B_0(1)$:

$$\omega_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

For instance,

$$\omega_1 = 2, \qquad \omega_2 = 2\pi, \qquad \omega_3 = 4\pi, \qquad \omega_4 = 2\pi^2.$$

Similarly we define the averages of the initial data:

$$G(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_x(r)} g(y)\, dS_y,$$

$$H(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_x(r)} h(y)\, dS_y,$$

Remark 1. We recall that the volume of the unit ball $|B_0(1)|$ and the area of the unit surface $|\partial B_0(1)|$ are related by

$$|B_0(1)| = \frac{\omega_n}{n}$$

This fact is, in principle, well known but we can derive this relation immediately from the co-area formula. Indeed, from formula in Example 1 we have for $h(x) = 1$:

$$|B_0(r)| = \int_{B_0(r)} 1 dx = \int_0^r dt \int_{\partial B_0(\xi)} 1\, dS_y = \int_0^r \omega_n \xi^{n-1}\, d\xi = \frac{\omega_n}{n} r^n$$

Hence we obtain $|B_0(r)| = \frac{\omega_n}{n} r^n$ which yields the required relation for $r = 1$.

Lemma 1 (Euler-Poisson-Darboux equation). *Fix* $x \in \mathbb{R}^n$ *and let* u *satisfy the initial value problem above. Then* $U(x,t,r) \in C^2([0;+\infty)^2)$ *and*

$$U''_{tt} - U''_{rr} - \frac{n-1}{r} U'_r = 0, \qquad t > 0, \qquad r > 0$$

$$U(x,0,r) = G(x,r), \qquad U'_t(x,0,r) = H(x,r), \quad r > 0$$

Proof. The fact that $U(x,t,r)$ is two times continuously differentiable in t follows from standard facts on differentiability of an integral depending on parameters. The situation with parameter r requires however a further analysis because the set $B_x(r)$ itself depends on r.

Step 1: We show that $U(x,t,r)$ is continuously differentiable in r and

$$U'_r(x,t,r) = \frac{1}{\omega_n r^{n-1}} \int_{B_x(r)} \Delta_y u(y,t) \; dy.$$

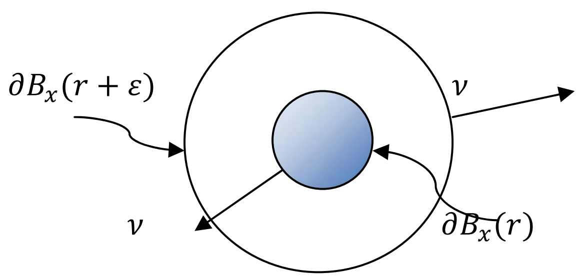
We consider an auxiliary function

$$f(r) = \omega_n r^{n-1} U(x,t,r) = \int_{\partial B_x(r)} u(y,t) \, dS_y$$

and for $\varepsilon > 0$ we have

$$f(r+\varepsilon) - f(r) = \int_{\partial B_x(r+\varepsilon)} u(y,t) \, dS_y - \int_{\partial B_x(r)} u(y,t) \, dS_y = \int_{\partial V} u(y,t) \, dS_y$$

where $V \coloneqq B_x(r+\varepsilon) \setminus \overline{B_x(r)}$ is the spherical slab with the oriented boundary (equipped with outward normals): $\partial V = \partial B_x(r+\varepsilon) - \partial B_x(r)$, see the picture below:



In order to apply the divergence theorem to the latter integrals we observe the following identity. From Example 1 above we know that unit vector-field $F(y) := \frac{y-x}{|y-x|}$ coincides with the gradient of the distant function:

$$F(y) = \frac{y-x}{|y-x|} = \nabla_y |y-x|.$$

For the further convenience we notice that the divergence of this vector-field is

$$\mathrm{div}_y F(y) = \mathrm{div}_y \frac{y-x}{|y-x|} = \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\left(\frac{y_k - x_k}{|y-x|}\right) = \frac{n}{|y-x|} - \sum_{k=1}^{n} \frac{(y_k - x_k)^2}{|y-x|^3} = \frac{n-1}{|y-x|}.$$

On the other hand, the unit normal vector at $y \in \partial B_x(s)$ is found as

$$\nu = \frac{y-x}{s} \equiv F(y).$$

In particular, we have for the scalar products: $\nu \cdot F(y) = \nu \cdot \nu = F(y) \cdot F(y) = 1$. Summarizing this and applying the divergence theorem we obtain:

$$f(r+\varepsilon) - f(r) = \int_{\partial V} 1 \cdot u(y,t)\, dS_y = \int_{\partial V} \nu \cdot F(y) u(y,t)\, dS_y = \int_V \mathrm{div}_y\, (F(y)u(y,t))\, dy$$

Applying the formula from Example 1 again, we see that

$$f(r+\varepsilon) - f(r) = \int_r^{r+\varepsilon} d\xi \int_{\partial B_x(\xi)} \mathrm{div}_y\, (F(y)u(y,t))\ dS_y$$

and it follows easily that the following limit exists

$$\lim_{\varepsilon \to 0+} \frac{f(r+\varepsilon) - f(r)}{\varepsilon} = \int_{\partial B_x(r)} \mathrm{div}_y\, (F(y)u(y,t))\ dS_y$$

Similarly the left limit is shown to exist, hence $f(r)$ is (continuously) differentiable and

$$f'(r) = \int_{\partial B_x(r)} \mathrm{div}_y\, (F(y)u(y,t))\ dS_y.$$

In order to transform the latter integral, we apply the above formula for div $F(y)$ and recall again that the unit normal ν coincides with $F(y)$:

$$\mathrm{div}_y \left(F(y)u(x,t)\right) = u(x,t)\, \mathrm{div}_y\, F(y) + F(y) \cdot \nabla_y u(y,t) = u(y,t) \frac{n-1}{|y-x|} + \nu \cdot \nabla_y u(y,t)$$

Hence applying the divergence theorem we get

$$f'(r) = \int_{\partial B_x(r)} \left(u(y,t) \frac{n-1}{|y-x|} + \nu \cdot \nabla_y u(y,t) \right) dS_y = \frac{n-1}{r} f(r) + \int_{B_x(r)} \Delta_y u(y,t)\, dy.$$

On the other hand,

$$f'(r) - \frac{n-1}{r} f(r) = r^{n-1}(r^{1-n} f(r))'$$

Returning to the definition of $U(x,t,r)$ we obtain

$$\int_{B_x(r)} \Delta_y u(y,t)\, dy = r^{n-1}(r^{1-n} f(r))' \equiv\ \omega_n r^{n-1} U_r'(x,t,r)\,.$$

It follows also from the above argument that differentiability of $f'(r)$ is equivalent to that of integral $\int_{B_x(r)} \Delta_y u(y,t)\, dy$. But this integral is easily seen to be differentiable (see again formula in Example 1) because the Laplacian $\Delta_y u(y,t)$ is a continuous function in all parameters.

Step 2. We return to our formula for $U_r'(x,t,r)$ and apply to it our wave equation:

$$\omega_n r^{n-1} U_r'(x,t,r) = \int_{B_x(r)} \Delta_y u(y,t)\, dy = \int_{B_x(r)} u_{tt}''(y,t)\, dy$$

Differentiation of the latter identity w.r.t. r yields by the co-area formula (see Example 1)

$$\left(\omega_n r^{n-1} U_r'(x,t,r)\right)_r' = \frac{\partial}{\partial r}\int_{B_x(r)} u_{tt}''(y,t)\, dy = \int_{\partial B_x(r)} u_{tt}''(y,t)\, dy = \frac{\partial^2}{\partial t^2}\int_{\partial B_x(r)} u(y,t)\, dy$$

which implies the desired equation:

$$(r^{n-1}U_r')_r' = (r^{n-1}U)_{tt}'' = r^{n-1}U_{tt}''.$$

The initial conditions $U(x,0,r) = G(x,r)$ and $U_t'(x,0,r) = H(x,r)$ follow immediately from the definition. ■

Casen=3

The idea how the above method works is the following simple relation

$$u(x,t) = \lim_{r\to 0+} U(x,r,t)$$

which makes it possible to connect any solution of the symmetrized equation to the corresponding solution of the initial wave equation. It turns out however that the method works differently for even and odd dimensions. Below we describe briefly the 3-dimensional case.

We suppose that $u(x,t) \in C^2(\mathbb{R}^3 \times [0,+\infty))$ solves the following initial value problem:

$$u_{tt}'' = u_{x_1x_1}'' + u_{x_2x_2}'' + u_{x_3x_3}''$$

$$u(x,0) = g(x), \quad u_t'(x,0) = h(x), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

We set additionally

$$\widetilde{U} = rU, \quad \widetilde{G} = rG, \quad \widetilde{H} = rH$$

Lemma 2. *Function $\widetilde{U}$ solves the following two-dimensional initial value problem*

$$\widetilde{U}_{tt}'' - \widetilde{U}_{rr}'' = 0, \qquad r > 0, \qquad t > 0$$

$$\widetilde{U}(x,0,r) = \widetilde{G}(x,r), \qquad \widetilde{U}_t'(x,0,r) = \widetilde{H}(x,r)$$

$$\widetilde{U}(x,t,0) = 0, \qquad t > 0.$$

Proof. Indeed, in our case $n = 3$, hence

$$\widetilde{U}''_{tt} = rU''_{tt} = r^{-1}(r^2U'_r)'_r = rU''_{rr} + 2U'_r = (U + rU'_r)'_r = \widetilde{U}''_{rr}$$

Applying the reflection principle (see formula on page 4 in Lecture 7) we find for $0 \le r \le t$:

$$\widetilde{U}(x,r,t) = \frac{1}{2}\left[\tilde{G}(x,t+r) - \tilde{G}(x,t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \widetilde{H}(x,s)\, ds.$$

Since $u(x,t) = \lim\limits_{r\to 0+} U(x,r,t)$, we find that

$$u(x,t) = \lim_{r\to 0+} \frac{\widetilde{U}(x,r,t)}{r} = \lim_{r\to 0+} \left(\frac{1}{2r}\left[\tilde{G}(x,t+r) - \tilde{G}(x,t-r)\right] + \frac{1}{2r}\int_{t-r}^{t+r} \widetilde{H}(x,s)\, ds \right)$$
$$= \tilde{G}'_t(x,t) + \widetilde{H}(x,t).$$

Hence we arrive at the *Kirchhoff formula*

$$u(x,t) = \frac{\partial}{\partial x}\left(\frac{1}{4\pi t}\int_{\partial B_x(t)} g(y)\, dS_y\right) + \frac{1}{4\pi t}\int_{\partial B_x(t)} h(y)\, dS_y$$

Case n=2

We give only the final result, the so-called Poisson formula. Namely, in the above notation,

$$u(x,t) = \frac{1}{4\pi t}\int_{\partial B_x(t)} \frac{tg(y) + t^2\, h(y) + t\, (\nabla g(y) \cdot (y - x))}{1}\, dS_y,$$

where $x \in \mathbb{R}^2$ and $t > 0$.