Lecture 8: The Laplace and Poisson equations

The Laplace equation

$$\Delta u(x) \equiv u''_{x_1x_1} + u''_{x_2x_2} + \cdots + u''_{x_nx_n} = 0$$

where $x = (x_1, x_2, \dots, x_n) \in \Omega$ and Ω is a domain in $\mathbb{R}^n$. Solutions of this equation are called *harmonic functions.*

The Poisson equation is a non-homogeneous Laplace equation:

$$\Delta u(x) = f(x), \qquad x \in \Omega$$

- Both equations are of elliptic type (there are no characteristics)

- Field theory: the function $-f(x)/\omega_n$ is the density of masses distributed in the body Ω produced the potential $u(x)$ inside the body. Outside the body, the potential is a harmonic function – a solution of a homogeneous Laplace equation.

- Similarly, one can consider a potential electric field with the gradient $u(x)$ and the charge density $-f(x)/\omega_n$.

- The well-known *inverse problem in potential theory* asks whether there exist two different bodies in $\mathbb{R}^3$ having the same potentials outside the union of bodies.

- Theory of functions of one complex variable deals with the "complexified" harmonic functions, the so-called holomorphic functions.

Example 1. Separation of variables. Consider the two-dimensional Laplacian:

$$u''_{xx} + u''_{yy} = 0$$

in some rectangle $\Omega = \{(x, y)\colon\ 0 \le x \le a,\ 0 \le y \le b\}$ with the boundary conditions:

$$u(0, y) = u(a, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, b) = g(x),$$

where $0 \le x \le a,\ 0 \le y \le b$.

Then it is natural to study solutions with horizontal-vertical symmetry, i.e. the solutions which admit separation of variables in the following sense:

$$u(x, y) = v(x)w(y).$$

This implies $v''(y)w(y) + v(x)w''(y) = 0$, and due to independency of x and y we find that

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = C$$

where C is some constant. Then our form of solution yields $v(0) = v(a) = w(0) = 0$.

Since $v'' - Cv = 0$ and $v(0) = v(a) = 0$, in order to avoid trivial case we have to assume that $C < 0$, say $C = -c^2$. Then $v'' + c^2 v = 0$ gives a series of solutions with zero-boundary condition:

$$v_n(x) = \sin\frac{\pi n x}{a}, \qquad n = 1,2,3, \dots$$

with the corresponding $c_n = \frac{\pi n}{a}$. Hence for the second component we obtain $w'' - c_n^2 w = 0$, which yields

$$w(y) = Ae^{c_n y} + Be^{-c_n y}$$

And after substitution of the boundary conditions we obtain $B = -A$.

Now we assume that our solution can be found as a formal sum of the simplest solutions found above, that is

$$u(x,y) = \sum_{n=1}^{\infty} v_n(x) w_n(y)$$

Then applying our initial conditions we find

$$g(x) = \sum_{n=1}^{\infty} v_n(x) w_n(b) = \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{a}\,(e^{c_n b} - e^{-c_n b}) = \sum_{n=1}^{\infty} 2A_n \sin\frac{\pi n x}{a}\sinh\frac{\pi n b}{a}$$

The coefficients A_n then can be found by expanding the function $g(x)$ into Fourier series and the solution takes the form

$$u(x,y) = \sum_{n=1}^{\infty} 2A_n \sin\frac{\pi n x}{a}\sinh\frac{\pi n y}{a}$$

Remark. Convergence of the above series for continuous $g(x)$ is verified as follows. Let $g(x) = \sum_{n=1}^{\infty} g_n \sin\frac{\pi n x}{a}$ be the Fourier expansion of $g(x)$. Then $A_n = \frac{g_n}{2\sinh\frac{\pi n b}{a}} = O(1)$. Moreover we have

$$\left|\frac{\sinh\frac{\pi n y}{a}}{\sinh\frac{\pi n b}{a}}\right| \sim e^{-\frac{n(b-y)}{a}}, \quad \text{as } n \to \infty.$$

Hence the series for $u(x,y)$ (with all derivatives) converges uniformly in any rectangle $0 \le x \le a$, $0 \le y \le b_1$ for any $b_1 < b$.

Example 2. Let $\Omega = \{(x,y)\colon\ x^2 + y^2 < 1\}$ be the unit disk. Consider the following problem:

$$u''_{xx} + u''_{yy} = 0, \quad (x,y) \in \Omega,$$

$$\frac{\partial u}{\partial \nu} = h \qquad \text{on } \partial\Omega\,.$$

Here $h = h(\theta)$ is some continuous function on the unit circle (where θ is the polar angle, i.e. $\tan\theta = \frac{y}{x}$). Rewriting the Laplacian in the polar coordinates yields (verify this!)

$$\Delta u = u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta} = 0$$

for $0 \le \theta \le 2\pi$ and $0 \le r < 1$. Here $r = \sqrt{x^2 + y^2}$ is the polar radius. The boundary condition takes then the form[1]

$$\frac{\partial u}{\partial \nu} \equiv u'_r|_{r=1} = h(\theta).$$

The new change of variables, $r = e^{-t}$, readily yields

$$r^2\Delta u = r^2 u''_{rr} + r u'_r + u''_{\theta\theta} = u''_{tt} + u''_{\theta\theta} = 0$$

Now we can apply the method of separation of variables described in Example 1. We assume that our solution can be represented as a sum of simple products

$$u(x, y) = v(t)w(\theta).$$

One of the factors, namely $w(\theta)$, must be 2π-periodic, because we are looking for a *continuous* solution. Thus

$$v''(t)w(\theta) + v(t)w''(\theta) = 0$$

and separating of variables yields

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = C.$$

The constant C must be positive to ensure that $w(\theta)$ is a *periodic* function. Taking into account that $w(\theta)$ is actually 2π-periodic, we obtain

$$w(\theta) = w_n(\theta) = A_n \sin\, n\theta + B_n \cos n\,\theta\,, \qquad n = 0,1,2, \ldots$$

Hence $C = n^2$ and this yields the equation for the remaining factor:

$$v''(x) - n^2 v(x) = 0,$$

which provides the series of (exponential) solutions:

$$v(t) = v_n(t) = a_n \sinh\, nt + b_n \cosh nt\,, \qquad n = 0, 1, 2, \ldots$$

Since $\sinh\, nt = \frac{1}{2}(e^{nt} - e^{-nt}) = \frac{1}{2}(r^n - r^{-n})$ and $\cosh\, nt = \frac{1}{2}(r^n + r^{-n})$, we arrive finally at

$$u_n(x, y) = v_n(t)w_n(\theta) = (c_n r^n + d_n r^{-n})(A_n \sin\, n\,\theta + B_n \cos n\,\theta).$$

On the other hand, taking into account that the desired solution must be continuous in the unit disk, we conclude that all $d_n = 0$, hence

[1] $\frac{\partial u}{\partial \nu} = u'_x \cos\theta\, + u'_y \sin\theta = u'_r$

$$u_n(x,y) = r^n(C_n \sin\, n\,\theta + D_n \cos n\,\theta).$$

This shows that our solution must be the following series:

$$u(x,y) = \sum_{n=0}^{\infty} r^n(C_n \sin\, n\,\theta + D_n \cos n\,\theta). \qquad (*)$$

In order to recover the undefined coefficients we apply the boundary condition:

$$u'_r|_{r=1} = \sum_{n=0}^{\infty} nr^{n-1}(C_n \sin\, n\,\theta + D_n \cos n\,\theta)\,|_{r=1} = \sum_{n=1}^{\infty} nC_n \sin\, n\,\theta + nD_n \cos n\,\theta = h(\theta)$$

Hence nC_n and nD_n are the Fourier coefficients of $h(\theta)$. Notice that it follows from the above formula that $h(0) = 0$. Finally, applying Abel's theorem, we conclude that the series $(*)$ converges in the open unit disk.

Some further remarks

- We encounter here a new phenomenon: as it is seen from the previous examples, the Laplace equation in a bounded domain may be *overdetermined*, that is one can not specify both $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ along the boundary.

- The first type of boundary problem, $u|_{\partial\Omega} = g$, is called the *Dirichlet* problem, the second, $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = h$, is called the *Neumann* problem

- Sometimes one considers a mixed problem, by prescribing both the Dirichlet and the Neumann data on some pieces of the boundary.

Green's identities

First we notice two simple consequences of the divergence theorem for functions $u, v \in C^2(\overline{\Omega})$ (two times continuously differentiable up to the boundary), where Ω is a bounded open set in $\mathbb{R}^n$ with piecewise smooth boundary. Namely,

$$\int_{\partial\Omega} v\,\frac{\partial u}{\partial \nu}\,dS = \int_{\Omega} (v\,\Delta u + \nabla u\,\cdot \nabla v)\,dx$$

and its skew-symmetric analogue:

$$\int_{\partial\Omega} v\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial v}{\partial \nu}\,dS = \int_{\Omega} (v\,\Delta u - u\,\Delta v)\,dx.$$

Now we list some important corollaries of the Green identities.

- ***Uniqueness***. Let us assume that u is a harmonic function. Substituting $v = u$ into the first Green's identity implies

$$\int_{\partial\Omega} u\,\frac{\partial u}{\partial \nu}\,dS = \int_{\Omega} |\nabla u|^2\,dx.$$

Observe that the latter integral is strictly positive unless u is a constant. Therefore in the left hand side the product $u\,\frac{\partial u}{\partial \nu}$ can't be identically zero on $\partial\Omega$ unless u is a constant. This yields immediately that the Dirichlet problem determines a solution uniquely while the Neumann problem determines the solution uniquely up to an additive constant.

- ***Necessary condition for the Neumann condition.*** Let u be a harmonic function with the Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = h$. Then taking $v = 1$ in the first Green's identity we obtain

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS = 0,$$

hence the average of h over the boundary equals zero: $\int_{\partial\Omega} h\ dS = 0$. In particular, h can't be chosen arbitrarily.

The fundamental solution

Let us attempt to find a solution u of the Laplace equation in $\mathbb{R}^n$ having the radial symmetric form:

$$u(x) = v(r),$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. We have the following idenitites: $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$,

$$\frac{\partial u}{\partial x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_i} = \frac{v''(r)x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), \qquad i = 1,2,\dots,n$$

Therefore

$$\Delta u = v''(r) + \frac{n-1}{r}\, v'(r) = 0$$

and integrating this identity gives $v(r) = a\ln r + b$ for $n = 2$ and $v(r) = \frac{a}{r^{n-2}} + b$ for and for $n \geq 3$.

Definition. The function

$$\Psi(x) = \begin{cases} \frac{1}{2\pi}\ln|x| & n = 2 \\ -\frac{1}{(n-2)\omega_n |x|^{n-2}} & n \geq 3 \end{cases}$$

is called the *fundamental solution* of the Laplace equation.

Theorem 1 (*Mean-value property*). *Let u be a harmonic function in the closed disk $B_a(R)$. Then*

$$\frac{1}{|\partial B_a(R)|}\int_{\partial B_a(R)} u\,(x)\; dS_x = \frac{1}{|B_a(R)|}\int_{B_a(R)} u\,(x)\; dx = u(a).$$

Proof. We consider here the case $n \geq 3$. The two-dimensional case is treated similarly. Moreover, without loss of generality we can assume that $a = 0$.

We consider $R > \varepsilon > 0$ and take

$$v(x) = \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \equiv w(|x|), \qquad w(t) = \frac{1}{t^{n-2}} - \frac{1}{R^{n-2}}$$

in the second Green's identity with Ω to be the spherical slab $V = B_0(R)\backslash\overline{B_0(\varepsilon)}$. Then $v(x)$ is harmonic in V, radially symmetric and vanishes on the exterior boundary: $v(x) = 0$ on $\partial B_0(R)$.

Therefore

$$0 = \int_{\partial V} (v\,\frac{\partial u}{\partial \nu} - u\,\frac{\partial v}{\partial \nu})\; dS_x$$

$$= -w(\varepsilon)\int_{\partial B_0(\varepsilon)} \partial_\nu u\; dS_x - \partial_r w(R)\int_{\partial B_0(R)} u\,dS_x + \partial_r w(\varepsilon)\int_{\partial B_0(\varepsilon)} u\,dS_x$$

where we used the decomposition $\partial V = \partial B_0(R) - \partial B_0(\varepsilon)$ (all the boundaries are assumed to be oriented by the outer normal) and the following equality for the normal derivative along the sphere of radius $t = |x|$: $\partial_r w(t) = \partial_\nu v(x)$. We find also

$$\partial_r w(R) = -\frac{n-2}{R^{n-1}}, \qquad \partial_r w(\varepsilon) = -\frac{n-2}{\varepsilon^{n-1}}$$

Hence

$$\left(\frac{1}{R^{n-2}} - \frac{1}{\varepsilon^{n-2}}\right)\int_{\partial B_0(\varepsilon)} \partial_\nu u\; dS_x = -\frac{n-2}{R^{n-1}}\int_{\partial B_0(R)} u\,dS_x + \frac{n-2}{\varepsilon^{n-1}}\int_{\partial B_0(\varepsilon)} u\,dS_x.$$

Notice that $\int_{\partial B_0(\varepsilon)} \partial_\nu u\; dS_x = O(\varepsilon^{n-1})$ for small ε, hence the left hand side of the displayed identity converges to zero as $\varepsilon \to 0+$. Since $|\partial B_0(R)| = \omega_n R^{n-1}$, it follows then

$$\frac{1}{\omega_n R^{n-1}}\int_{\partial B_0(R)} u\,dS_x = \lim_{\varepsilon\to 0}\frac{1}{\omega_n \varepsilon^{n-1}}\int_{\partial B_0(\varepsilon)} u\,dS_x = u(0)$$

and the first identity follows. The second identity is obtained by integrating the first one:

$$\frac{\omega_n R^n u(0)}{n} = \int_0^R u(0)t^{n-1}dt = \int_0^R dt\int_{\partial B_0(t)} u\,dS_x = \int_{B_0(t)} u\,dx$$

and observing that $|B_0(R)| = \frac{\omega_n R^n}{n}$. ■

Converse to mean-value property. The mean-value property is also used as an equivalent definition of harmonicity. In fact, if function u is defined in a domain Ω, integrable there and for any ball $B_x(t) \subset \Omega$ satisfies the mean value property

$$u(x) = \frac{1}{|B_x(t)|} \int_{B_x(t)} u\,(y)\ dy$$

then u is harmonic in Ω.

Dirichlet data. The mean-value property can also be interpreted as a solution of the Diriclet problem $u(x) = g(x),\ x \in \partial\Omega$, at the origin ($u$ is a harmonic function). Indeed, we have

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_a(R)} g\,(x)\ dS_x$$

Theorem 2 (Strong maximum principle). *Suppose* $u(x) \in C^2\,(\Omega) \cap C^0(\bar{\Omega})$ *is harmonic within a connected bounded open set* Ω*. Then*

$$\max_{\Omega\cup\partial\Omega} u(x) = \max_{\partial\Omega} u(x)$$

Furthermore, if there is a point $x_0 \in \Omega$ *such that* $u(x_0) = \max_\Omega u(x)$*, then* $u \equiv const.$

Proof. The second statement follows immediately from the mean-value property: if there exists a point $x_0 \in \Omega$ satisfying $u(x_0) = \max_\Omega u(x) =: M$, consider a small ball $B_{x_0}(r) \in \Omega$, then

$$M = u(x_0) = \frac{1}{|B_{x_0}(r)|} \int_{B_{x_0}(r)} u\,(y)\ dy \leq M$$

Hence $u \equiv M$ within $B_{x_0}(r)$. In other words, the set of points satisfying $u(x_0) = \max_\Omega u(x)$ is open. It is also easily seen that the set is closed. Hence, by connectedness of Ω, $u \equiv M$.

Corollary 1. (Uniqueness) *Let* $g(x) \in C^0\,(\partial\Omega)$ *and* $f(x) \in C^0\,(\Omega)$*. Then there exists at most one solution* $(x) \in C^2\,(\Omega) \cap C^0(\bar{\Omega})$ *of the boundary-value problem*

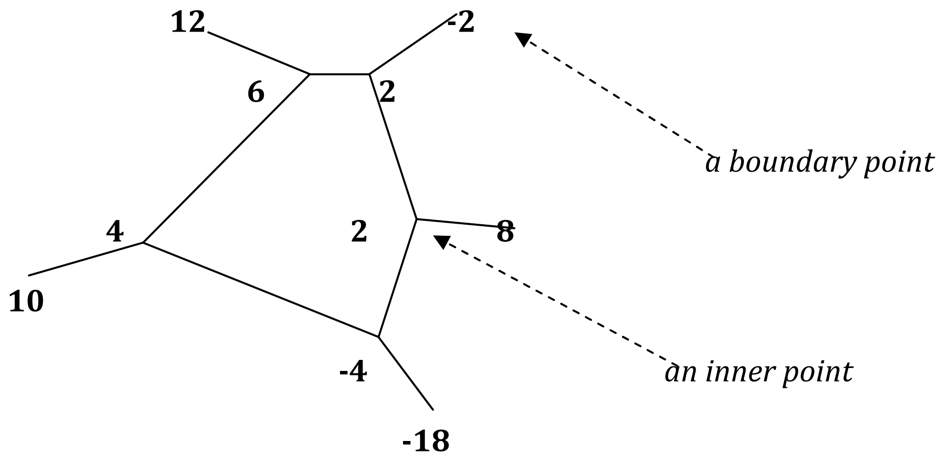
$$\Delta u(x) = f(x), \qquad x \in \Omega,$$

$$u(x) = g(x), \qquad x \in \partial\Omega.$$

Interlude: *Dirichlet problem on planar graphs*

The mean-value property allows to define harmonic functions on such exotic objects as graphs. We briefly explain this below.

Let $G = (V, E)$ be a connected finite graph, where V is the set of vertices and $E \subset V \times V$ is the set of edges. A function on a graph is a map $f: V \to \mathbb{R}$. Two vertices x and y are adjacent if $(x, y) \in E$.



Picture*: An example of harmonic graph*

A function on a graph is called harmonic if it satisfies the mean-value property:

$$f(x) = \frac{1}{N_x} \sum f(y)$$

where the sum is taken over all vertices y adjacent to x and N_x is the number of these y.

The Dirichlet problem: find a harmonic function on a graph by given "boundary values". This leads to interesting interplay with linear algebra, graph theory and potential theory.

Theorem 3. *Let $u(x)$ be a twice continuously differentiable function with compact support, $u(x) \in C_0^2(\mathbb{R}^n)$. In other words, $u(x) = 0$ outside some compact set in $\mathbb{R}^n$. Then*

$$u(0) = \int_{\mathbb{R}^n} \Psi(x)\, \Delta u(x)\, dx$$

Proof. Let $u(x) = 0$ for $|x| \geq R$ and let $B_0(R)$. Notice that $\Psi(x)\, \Delta u(x)$ is integrable in $B_0(R)$. Indeed, $\Delta u(x)$ is continuous, hence bounded. The only singular point for $\Psi(x)$ is the origin. But there it is integrable because $\ln|x|$ is integrable for $n = 2$ and $|x|^{-p}$ is integrable in $B_0(R)$ for $p < n - 1$ (why?).

Therefore, removing a small ball centered at the origin, $V_\varepsilon = B_0(R) \backslash \overline{B_0(\varepsilon)}$, we obtain

$$\int_{\mathbb{R}^n} \Psi\, \Delta u\, dx = \int_{B_0(R)} \Psi\, \Delta u\, dx = \lim_{\varepsilon \to 0+} \int_{V_\varepsilon} \Psi\, \Delta u\, dx$$

The latter integral may be treated as in Theorem 1 above, by noticing that $u = 0$ on $\partial B_0(R)$ together with its normal derivative there:

$$\int_{V_\varepsilon} \Psi\, \Delta u\, dx = \int_{V_\varepsilon} (\Psi\, \Delta u - u \Delta \Psi)\, dx = -w(\varepsilon) \int_{\partial B_0(\varepsilon)} \partial_\nu u\, dS_x + \partial_r w(\varepsilon) \int_{\partial B_0(\varepsilon)} u\, dS_x$$

where $\Psi(x) = w(|x|)$ and

$$w(r) = \begin{cases} \dfrac{1}{2\pi} \ln r & n = 2 \\ -\dfrac{1}{(n-2)\omega_n r^{n-2}} & n \geq 3 \end{cases}$$

Because $\partial_\nu u$ is bounded and $|\partial B_0(\varepsilon)| \sim \varepsilon^{2-n}$, we have

$$\lim_{\varepsilon \to 0+} w(\varepsilon) \int_{\partial B_0(\varepsilon)} \partial_\nu u\, dS_x = 0,$$

and since $\partial_r w(\varepsilon) = \frac{1}{\omega_n \varepsilon^{n-1}}$, we obtain finally

$$\lim_{\varepsilon \to 0+} \int_{V_\varepsilon} \Psi\, \Delta u\, dx = \lim_{\varepsilon \to 0+} \partial_r w(\varepsilon) \int_{\partial B_0(\varepsilon)} u\, dS_x = \lim_{\varepsilon \to 0+} \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_0(\varepsilon)} u\, dS_x = u(0)$$

and the result follows. ∎

Corollary 2 (Solving the Poisson equation). *Let $f(x)$ be twice continuously differentiable, and let $f(x) = 0$ outside some compact set in $\mathbb{R}^n$. Define*

$$u(x) = \int_{\mathbb{R}^n} \Psi(x - y) f(y) dy$$

Then $u(x) \in C^2(\mathbb{R}^n)$ and $\Delta u = f$ in $\mathbb{R}^n$.