Lecture 10: The Laplace equations cont.

Recall that the function

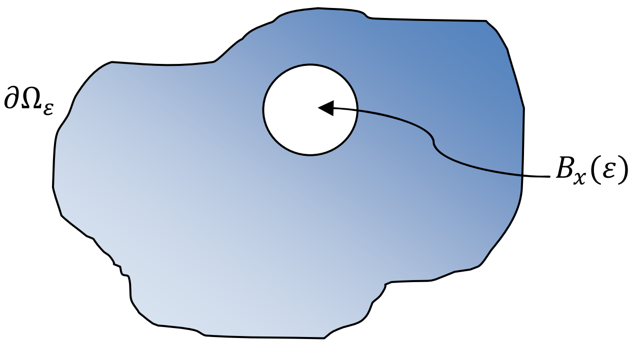
$$\Psi(x) = \begin{cases} \frac{1}{2\pi}\ln|x| & n = 2 \\ -\frac{1}{(n-2)\omega_n |x|^{n-2}} & n \geq 3 \end{cases}$$

is called the *fundamental solution* of the Laplace equation.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $u(x) \in C^2(\overline{\Omega})$ (twice continuously differentiable up to the boundary). Then*

$$u(x) = \int_\Omega \Psi(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} \big(u(y)\,\partial_\nu \Psi(x-y) - \Psi(x-y)\,\partial_\nu u(y)\big) dS_y$$

Proof. Let $x \in \Omega$ and let $B_x(2\varepsilon) \subset \Omega$. Consider $\Omega_\varepsilon = \Omega \setminus \overline{B_x(\varepsilon)}$



Then applying the second Green identity we get

$$\int_{\Omega_\varepsilon} \Psi(x-y)\,\Delta u(y)\,dy = \int_{\partial\Omega_\varepsilon} \big(u(y)\,\partial_\nu \Psi(x-y) - \Psi(x-y)\,\partial_\nu u(y)\big) dS_y$$

where $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_x(\varepsilon)$. Now let $\varepsilon \to 0$.

- Since Ψ is locally integrable, the integral over Ω_ε becomes the integral over Ω.

- The integral over the sphere $\partial B_x(\varepsilon)$ is (for $n \geq 3$)

$$\varepsilon^{n-1} \int_{|t|=1} \big(u(x+\varepsilon t)\,\partial_\nu \Psi(\varepsilon t) - \Psi(\varepsilon t)\,\partial_\nu u(x+\varepsilon t)\big) dS_t =$$
$$= \varepsilon^{n-1} \int_{|t|=1} \left(\frac{u(x+\varepsilon t)}{\omega_n \varepsilon^{n-1}} - \frac{1}{(n-2)\varepsilon^{n-2}}\,\partial_\nu u(x+\varepsilon t) \right) dS_t$$

Hence it converges to $u(x)$ (since the latter integral converges to zero) and we obtain the required formula. ■

Remark 1. Recall that in $\mathbb{R}^3$ the Coulomb's potential for a point charge placed at the origin (or the corresponding Newtonian potential) has the form

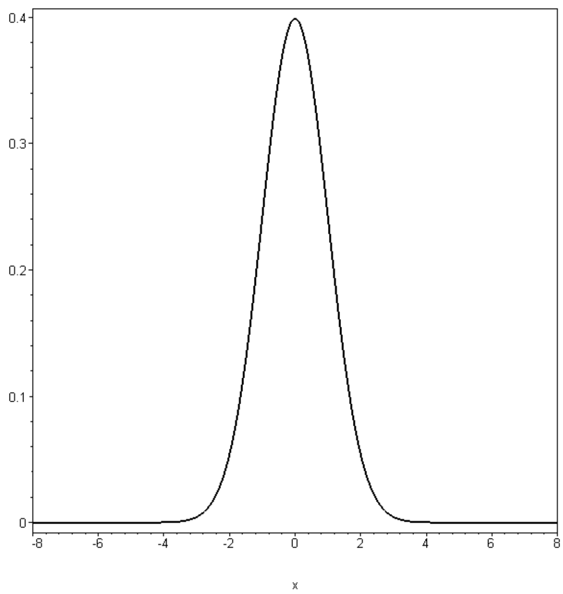
$$\frac{1}{|x|} = -4\pi\, \Psi(x)$$

Another way how to write this is

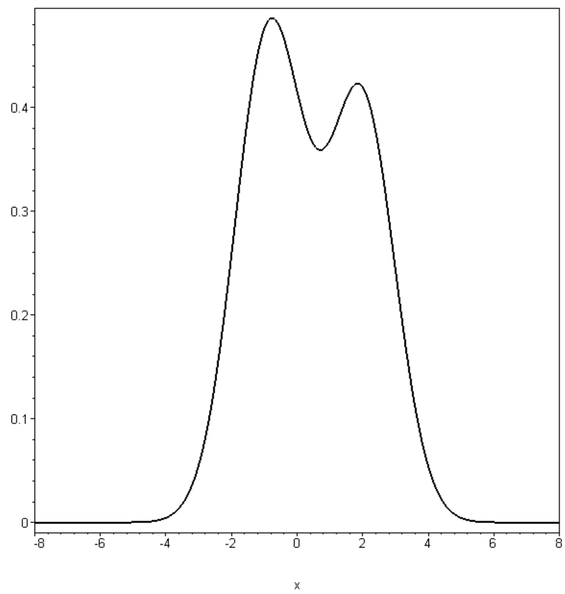
$$\frac{1}{|x|} = \int_{\mathbb{R}^3} \frac{\delta(y)}{|x-y|}\, dy = \ -4\pi \int_{\mathbb{R}^3} \Psi(x-y)\delta(y)\, dy$$

The latter may be interpreted as a weak form of our identity above for $u(x) = \frac{1}{|x|}$, where

$$\Delta u(x) = -4\pi\delta(y).$$



Delta-function $\boldsymbol{\delta(x)}$



An arbitrary density $\boldsymbol{\rho(x)}$

If we replace in the above integrals the singular density $\delta(x)$ by some continuous function wit compact support $\rho(x)$ (or rather $\rho(x) \in L^1(\mathbb{R}^3)$), we obtain a potential function induced by $\rho(x)$:

$$u_\rho(x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy = \ -4\pi \int_{\mathbb{R}^3} \Psi(x-y)\, \rho(y)\, dy$$

Notice that in this case we obtain the potential theoretic interpretation of the distribution of charges (masses) in terms of Laplacian:

$$\Delta u_\rho(x) = -4\pi\, \rho(x)$$

Corollary 1. *If $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary and $u(x) \in C^2(\overline{\Omega})$ then for any $\xi \in \Omega$ we have*

$$u(\xi) = \int_{\partial\Omega} \big(u(y)\, \partial_\nu \Psi(y - \xi) - \Psi(y - \xi)\, \partial_\nu u(y)\big) dS_y$$

Now we assume that for any point $\xi \in \Omega$ we can find a harmonic function $G(x, \xi)$ such that

- $G(x, \xi) - \Psi(x - \xi) =$ harmonic and continous in Ω
- $G(x, \xi) = 0$ on the boundary $\partial\Omega$

Definition. The function $G(x, \xi)$, if exists, is uniquely defined and it is called *Green's function* of the Laplacian for Ω (with respect to the Dirichlet boundary condition).

Notice that $G(x, \xi) = G(\xi, x)$

The main characteristic property of Green's function is the following integral formula

$$u(\xi) = \int_{\Omega} G(x, \xi)\, \Delta u(x)\, dx + \int_{\partial\Omega} u(x)\, \partial_\nu G(x, \xi)\, dS_y$$

which is valid for any $u(x) \in C^2(\overline{\Omega})$. In particular, for harmonic functions we obtain the representation by the boundary data:

$$u(\xi) = \int_{\partial\Omega} g(x) H(x, \xi)\, dS_y,$$

where $H(x, \xi) = \partial_\nu G(x, \xi)$ and $u|_{\partial\Omega} = g(x)$.

The Dirichlet problem in the ball (the Poisson formula)

The solution of the Dirichlet problem: $\Delta u(x) = 0$ in $B \equiv B_0(R)$, $u|_{\partial B} = g(x)$, is given by

$$u(\xi) = \frac{R^2 - |\xi|^2}{R\, \omega_n} \int_{|x|=R} \frac{g(x)}{|x - \xi|^n} dS_x$$