Lecture 11: Functional spaces

Let X be a *vector space* over some field F ($\mathbb{R}$ or $\mathbb{C}$) on which is defined a *norm*

$$\|\cdot\| : X \to \mathbb{R}^{\geq 0}.$$

That is $\|\cdot\|$ is a non-negative functional on X satisfying

- $\|a + b\| \leq \|a\| + \|b\|$
- $\|\lambda a\| = |\lambda| \cdot \|a\|$
- $\|a\| = 0$ if and only if $a = 0$

Any norm on X introduces the structure of metric space, actually, a normed vector space. Namely, the *distance* between two vectors a and b is defined as the number

$$d(a, b) = \|a - b\|$$

which is positive for distinct points and satisfies the axioms of metric spaces.

Frequently one defines the norm by virtue of a scalar product. We recall that a bilinear functional

$$\langle\cdot,\cdot\rangle : X \times X \to F$$

(F is either the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$) such that

- $\langle a, a\rangle$ is real and non-negative, and it vanishes if and only if $a = 0$.

Then the induced norm is given by

$$\|a\|^2 = \langle a, a\rangle.$$

(Note: a given norm by no means is generated by some scalar product.)

Cauchy-Schwarz inequality:

$$|\langle a, b\rangle| \leq \|a\| \, \|b\|$$

Topological definitions:

A metric space X is called *complete*, if any Cauchy sequence converges to a limit in the space. A complete vector space is called a *Banach space*.

A vector space X equipped with a scalar product $\langle\cdot,\cdot\rangle$ and complete as a metric space in the norm generated by $\langle\cdot,\cdot\rangle$, is called a *Hilbert space*. Usually one consider only Hilbert spaces which have a countable basis.

Basis and systems

Recall that a (at most countable) collection of vectors $\{x_k\}_{k\in A}$ is called orthogonal if for any two indices $k, j \in A$ and $k \neq j$:

$$\langle x_k, x_j \rangle = 0$$

This system is called orthonormal if additionally all vectors have the unit norm: $\|x_k\| = 1$ for all $k \in A$.

An orthonormal system $\{x_k\}_{k\in A}$ is called *complete* if for any $x \in X$:

$$x = \sum_{k\in A} \langle x, x_k \rangle\, x_k$$

(if A is infinite then the last series is convergent).

A complete, orthonormal set $\{x_k\}_{k\in A}$ is called an *orthonormal basis* for X, and the cardinality of A is called the dimension of X.

Example 1. Trigonometric series on the unit circle with bounded l^2-norm:

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{kx} : \qquad \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$$

The scalar product is defined as follows

$$\langle \sum_{k=-\infty}^{\infty} c_k e^{kx}, \sum_{k=-\infty}^{\infty} d_k e^{kx} \rangle = \sum_{k=-\infty}^{\infty} c_k \overline{d_k} < \infty.$$

This is the so-called Hardy space H^2 on the unit circle $\mathbb{R}/2\pi$. Any such series generates a holomorphic function in the unit disk and the scalar product may be interpreted as usual integral of product of two functions.

Example 2. The space P of all polynomials in $x_1, x_2, \ldots, x_n$ with real coefficients is a vector space with a natural stratification (homogeneous polynomials of a fixed degree). This space admits many scalar products, for example, the canonic scalar product is

$$\langle f, g \rangle = \int_{|x|=1} f(x) g(x)\; dS_x,$$

where the integral is taken over the unit sphere in $\mathbb{R}^n$. The following (finite) decomposition holds: for any polynomial $f \in P$

$$f = f_N + f_{N-2}|x|^2 + f_{N-4}|x|^4 + \cdots$$

Here f_k is a harmonic homogeneous polynomial of degree k and $|x|^2 = x_1^2 + \cdots + x_n^2$ is radially symmetric polynomial.

Example 3. Let K be a bounded closed (=compact) set in $\mathbb{R}^n$. Then $C(K)$, the vector space of all continuous functions on K, is a Banach space with respect to the uniform norm

$$\|f\|_{C(K)} = \|f\|_\infty = \max_{x\in K}|f(x)|.$$

NOTE: not induced by a scalar product.

Let X be a Hilbert space. A **bounded functional** $f: X \to F$ is a linear map which satisfies the property: there is a constant C such that

$$|f(x)| \leq C\|x\|$$

for any vector x in X.

Riesz representation theorem: *Let X be a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$. Then for any bounded functional f there is a vector $y_f \in X$ such that*

$$\langle y_f, x\rangle = f(x), \qquad \forall x \in X.$$

Idea of the proof: the kernel $f = 0$ is a closed Hilbert subspace of X of co-dimension one. Take the orthogonal (one-dimensional) space and a unit vector in it, say, y. Then there is $f(y)y$ will be the desired vector.

Lebesgue integral and measure in $\mathbb{R}^n$

Measure theory initially provides a notion of length/area/volume of subsets of Euclidean spaces with natural additivity and translation invariance properties. This suggests a suitable class of *measurable* subsets is an essential prerequisite.

The Lebesgue measure μ is constructed in accordance with the following principles:

- the measure of a null-set is equal to zero;
- measurable sets form a σ-algebra;
- the measure is σ-additive and non-negative (the infinity value is allowed);
- a function f is called measurable if for any real c the set $\{x: f(x) < c\}$ is a μ-measurable set;

Then the Lebesgue integral is defined first for simplest, the so-called (nonnegative) indicator functions, and then extends on general signed functions.

The principle advantage with the Lebesgue integral is a wider class of “permitted” operations: limits, sums etc.

Example 4. For any open set D in $\mathbb{R}^n$ and any $p > 1$ we define the so-called L^p- norm as

$$\|f\|_{L^p(D)} = \|f\|_p = \int_D |f(x)|^p \, dx$$

The latter integral should be understood as the Lebesgue integral.

Note: the space of continuous functions with respect to this p-norm will not be complete for any $p > 1$. On the other hand, one can prove that for any continuous function in $\overline{D}$ there holds

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

In general, we have the essential supremum which is defined as

$$\|f\|_\infty = \inf \{M:\ \mu\{x: u(x) > M\} = 0\}$$

- For any open set D in $\mathbb{R}^n$ the space $L^2(D)$ is a Hilbert space.

Hölder inequality:

$$\|fg\|_1 \leq \ \|f\|_p \ \|g\|_q$$

Minkowski inequality: if $p > 1$ then

$$\|f + g\|_p \leq \ \|f\|_p + \|g\|_p$$

- For any open set D in $\mathbb{R}^n$ and any $p > 1$, the space $L^p(D)$ is a Banach space.

Lipschitz functions: f is called a Lipschitz function in a set D in $\mathbb{R}^n$ if there is a non-negative M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for any two vectors x, y in D.

Hölder functions: there is $\alpha < 1$ and a non-negative M such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for any two vectors x, y in D. α is called the Hölder exponent.

One can define the following semi-norm:

$$\|f\|_{C^{0,\alpha}(D)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$