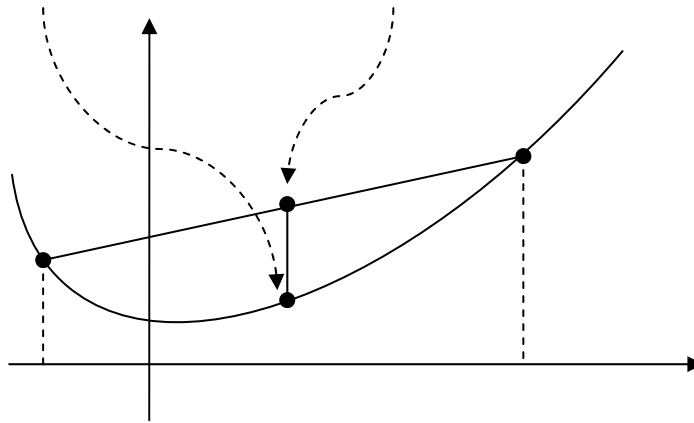


Maximum principles

Some motives:

- *Linear functions* on $[a, b] \subset \mathbb{R}^1$: solutions of differential equation $u''(x) = 0$
- *Convex functions*: if smooth then it is defined as a solution of inequality $u''(x) \geq 0$, otherwise satisfy the structural condition:

$$u(xt + y(1 - t)) \leq xu(x) + (1 - t)u(y), \quad \forall t \in [0,1], \quad \forall x, y \in [a, b]$$



For all these examples the following version of the **(weak) maximum principle** holds:

$$\sup_{x \in U} f(x) = \max_{x \in \partial U} f(x)$$

Moreover, for these examples the following **(strong) maximum principle** holds:

if $x_0 \in [a, b]$ is an inner point then

- either $f \equiv \text{const}$
- or $f(x_0) < \max_{x \in \partial U} f(x)$

Higher-dimensional analogues

- *linear functions* \rightarrow *harmonic functions* (solutions of $\Delta f(x) = 0$) satisfy the mean-value property

$$\frac{1}{|B_y(R)|} \int_{B_y(R)} f(x) dx = f(y), \quad \text{where } B_y(R) = \{x \in \mathbb{R}^n: |x - y| < R\}$$

in particular, the strong maximum principle also holds

- *convex functions* \rightarrow *subharmonic functions* (solutions of $\Delta f(x) \geq 0$) satisfy the mean-value property

$$\frac{1}{|B_y(R)|} \int_{B_y(R)} f(x) dx \geq f(y),$$

The maximum principle in the above sense depends on point-wise properties of a function, that is it requires some regularity the function. For example, it is meaningless for L^p -classes in the above form.

Maximum principles are also most appropriate for solutions of *elliptic* type PDEs, for instance, linear operators having the form

$$Lu = \sum_{i,j=1}^n a_{ij} u''_{x_i x_j} + \sum_{i=1}^n b_i u'_{x_i} + cu$$

In that case we always assume that all coefficients are (symmetric) continuous functions: $a_{ij}(x) = a_{ji}(x)$, $a_{ij}(x) \in C^0(\bar{U})$, $b_i(x), c(x) \in C^0(\bar{U})$ and $a_{ij}(x)$ satisfying the (uniform) *ellipticity condition*:

there is a constant $\alpha > 0$ such that for any $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha \|\xi\|^2.$$

Weak maximum principle

Theorem. Assume that $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U . If $Lu \geq 0$ in U then

$$\max_{x \in \bar{U}} u(x) = \max_{x \in \partial U} u(x).$$

Remark. A function satisfying $Lu \geq 0$ ($Lu \leq 0$ respectively) is called a supersolution (subsolution respectively).

Proof. Let us first suppose that we have a strict inequality

$$Lu > 0 \quad \text{in } U$$

and yet there exists a point $x_0 \in U$ with $\max_{x \in \bar{U}} u(x) = u(x_0)$. Then x_0 is an inner maximum point, that is we have the following necessary conditions:

$$Du(x_0) = 0$$

and

$$D^2u(x_0) \leq 0.$$

Since the matrix $A = (a_{ij}(x_0))_{i,j=1}^n$ is symmetric and positive definite, there exists an orthogonal matrix $C = (c_{ij})$ such that

$$\Lambda A \Lambda^T = \text{diag}(d_1, d_2, \dots, d_n), \quad d_l > 0.$$

Then we have for any $y = x_0 + B(x - x_0)$ we have

$$x - x_0 = \Lambda^{-1}(y - x_0) = \Lambda^T(y - x_0)$$

so that

$$u_{x_i} = \sum_{1 \leq k \leq n} u'_{y_k} \lambda_{ik}, \quad u''_{x_i x_j} = \sum_{1 \leq k, l \leq n} u''_{y_k y_l} \lambda_{ik} \lambda_{jl}$$

Hence at the point x_0 ,

$$\sum_{1 \leq i, j \leq n} a_{ij} u''_{x_i x_j} = \sum_{1 \leq k, l \leq n} \sum_{1 \leq i, j \leq n} u''_{y_k y_l} a_{ij} \lambda_{ik} \lambda_{jl} = \sum_{1 \leq k \leq n} d_k u''_{y_k y_k}$$

(since $\sum_{ij} a_{ij} \lambda_{ik} \lambda_{jl} = d_k \delta_{kl}$). Moreover, we have $d_k > 0$ and $u''_{y_k y_l}(x_0) \leq 0$, hence

$$Lu = \sum_{1 \leq i, j \leq n} a_{ij} u''_{x_i x_j} \leq 0$$

that contradicts our assumption $Lu > 0$. Thus, the first case is proved.

Now let a weak inequality $Lu \geq 0$ hold only. We introduce an auxiliary function

$$u_t(x) = u(x) + t e^{Mx_1}, \quad x \in U, M > 0.$$

The uniform ellipticity implies

$$a_{ii}(x) \geq \alpha > 0.$$

Therefore,

$$Lu_t = Lu + t L(e^{Mx_1}) \geq 0 + t e^{Mx_1} (M^2 a_{11} + b_1 M) \geq Mt e^{Mx_1} (M\alpha - \|b_1\|_\infty) > 0$$

in U for M large enough. According to previous step we have for any $t > 0$:

$$\max_{x \in \bar{U}} u_t(x) = \max_{x \in \partial U} u_t(x)$$

Now we let $t \rightarrow 0+$, so that we obtain as the limit relation

$$\max_{x \in \bar{U}} u_0(x) = \max_{x \in \partial U} u_0(x)$$

where $u_0 \equiv u$ and the required property follows.

Corollary. (Weak maximum principle for $c \leq 0$) Assume that $c \leq 0$ and $Lu \geq 0$ in U . Then

$$\max_{x \in \bar{U}} u(x) \leq \max_{x \in \partial U} u^+(x),$$

where $u^+ = \max\{u, 0\}$. If $Lu = 0$ then

$$\max_{x \in \bar{U}} |u(x)| \leq \max_{x \in \partial U} |u(x)|.$$

Proof. If $u \leq 0$ in U then our statement is trivial. Let us assume now that the set

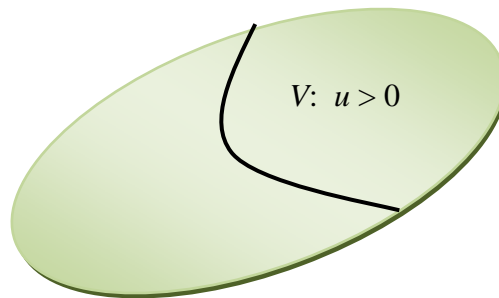
$$V = \{x \in U \mid u(x) > 0\}$$

is non-empty. Then

$$Ku = Lu - cu \geq 0, \quad x \in V.$$

On the other hand, we can apply the previous theorem to our new operator K since it contains no zero-term (c -term). We have (see picture)

$$\max_{x \in \bar{V}} u(x) = \max_{x \in \partial V} u(x) = \max_{x \in \partial U} u^+(x)$$



In order to prove the second statement one has to apply the first one to $-u$. ■

Strong maximum principle

Theorem (Hopf's lemma) *Assume that $u \in C^2(U) \cap C^1(\bar{U})$ and $c \equiv 0$ in U . Suppose further that $Lu \geq 0$ in U and that there exists a point $x_0 \in \partial U$ such that*

$$u(x_0) > u(x), \quad \text{for all } x \in U.$$

*Assume also that U is regular at x_0 in the sense that there exists an open ball $B \subset U$ with $x_0 \in \partial B$ (this condition is called **the interior ball condition**). Then*

- $\frac{\partial u}{\partial \nu}(x_0) > 0$ (ν is the outward unit normal to B at x_0)

Remark. A weak inequality $\frac{\partial u}{\partial \nu}(x_0) > 0$ is trivial and follows immediately from usual maximum property, so that significance of the theorem is the strict inequality $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Corollary (Strong maximum principle). *Let U be a connected open set and $u \in C^2(U) \cap C^1(\bar{U})$ be a solution of $Lu \geq 0$ in U with $c \equiv 0$ in U . If u attains its maximum over \bar{U} at some interior point, then $u \equiv \text{const}$ in U .*

Proof (of Corollary). Write $M = \max_{\bar{U}} u$ and consider the level set

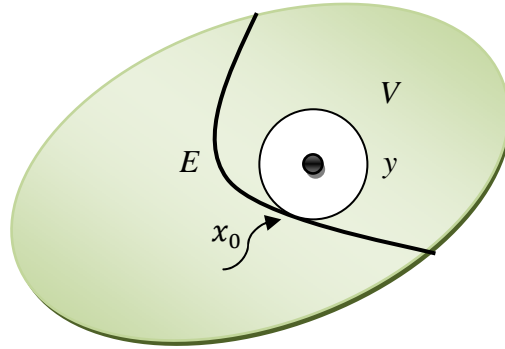
$$E = \{x \in U \mid u(x) = M\}.$$

Then if $u \not\equiv M$ then we can consider the set

$$V = \{x \in U \mid u(x) < M\}.$$

Choose a point $y \in V$ satisfying

$$\text{dist}(y, E) < \text{dist}(y, \partial U)$$



and denote by B the largest ball with center y whose interior lies in V . Then there exists some point $x_0 \in E \cap \partial B$. Clearly, V satisfies the interior ball condition at x_0 , whence Hopf's Lemma implies $\frac{\partial u}{\partial \nu}(x_0) > 0$. But This is a contradiction since u attains its maximum at x_0 (that is we have $Du(x_0) = 0$).