Solutions (Partiella differentialekvationer D, 2008-10-13, Vladimir Tkachev)

Problem1. (5 points) *Solve the Cauchy problem*

$$(u+x)u'_x-(u-y)u'_y=2u, \qquad u(3x,x)=2x.$$

Solution. The characteristic equations are

$$\dot{x}=u+y, \qquad \dot{y}=y-u, \qquad \dot{u}=2u$$

- The last equation yields $u=C_1e^{2t}$
- From the first equations we find: $\frac{d}{dt}(x+y)=(x+y)\Rightarrow x+y=C_2e^t$
- Similarly, we obtain:
  $$\frac{d}{dt}(x-y)=(x-y)+\frac{du}{dt}=(x-y)+2C_1e^{2t}$$
  and solving the last (linear) equation with respect to $x-y$ we find
  $$x-y=C_3e^t+2C_1e^{2t}$$
- Substitution the initial conditions and $t=0$ into the obtained system of equations yields
  $$x_0+y_0=C_2, \qquad x_0-y_0=C_3+2C_1, \qquad u_0=C_1$$
- The initial conditions are
  $$x_0=3s, \qquad y_0=s, \qquad u_0=2s$$
- Solving the linear system we find:
  $$C_1=2s, \quad C_2=4s, \quad C_3=-2s,$$
  that is
  $$u=2se^{2t}, \qquad x+y=4se^t, \qquad x-y=-2se^t+4se^{2t}$$
- Eliminating e^t we find $(x+y)+2(x-y)=8se^{2t}=4u$, and finally we obtain the required solution:
  $$u=\frac{3x-y}{4}$$

Problem 2. (6 points) *Solve the Cauchy problem for the nonlinear equation*

$$\left(\frac{\partial u}{\partial x}\right)^2+y\,\frac{\partial u}{\partial y}=u, \qquad u(x,1)=x^2.$$

Remark. The formulation of this problem on exam contained a misprint: in the left hand side, the partial derivative $\frac{\partial u}{\partial y}$ has been incorrectly displayed as $\frac{\partial u}{\partial x}$. The resulted problem becomes more trivial and it can be shown that it has no solution (below you can find the explanation of this and the solution of the planned problem). Criteria for this problem were changed: any correct and reasonable attempt to find solution by using the standard methods of solving of non-linear 1st order PDE was estimated by the maximal number of points (6). If an incorrect method or argument was used, the points were reduced respectively.

Solution of the actual problem. We assume that there is a solution to the following Cauchy problem

$$\left(\frac{\partial u}{\partial x}\right)^2 + y\,\frac{\partial u}{\partial x} = u, \qquad u(x,1) = x^2.$$

Then differentiating the initial data with respect to x we find

$$\frac{\partial u}{\partial x}(x,1) = 2x.$$

Inserting this relation together with the initial conditions: $x = x, y = 1$ and the initial solution $u(x,1) = x^2$ into the original equation yields

$$(2x)^2 + 1 \cdot\ 2x = x^2$$

However the latter identity holds only for two points $x = 0$ and $x = -2$ and it fails for any other x. This proves that the posed Cauchy problem has no solutions.

Solution of the planned problem. The equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + y\,\frac{\partial u}{\partial y} = u$$

has the form $F = p^2 + yq - u = 0$, so that the characteristic system is found as

$$\dot{x} = F'_p = 2p, \qquad \dot{y} = F'_q = y, \qquad \dot{u} = 2p^2 + yq = u + p^2$$

and similarly

$$\dot{p} = -F'_x - F'_u p = p, \qquad \dot{q} = -F'_y - F'_u q = q - q = 0$$

We find step-by-step:

- $p = p_0 e^t$
- $q = q_0$
- $y = y_0 e^t$
- $x = 2p_0 e^t + C_1$
- $u = C_2 e^t + p_0^2 e^{2t}$

The initial conditions are

$$x_0 = s, \qquad y_0 = 1, \qquad u_0 = s^2, \qquad p_0^2 + y_0 q_0 = u_0$$

and the strip condition is $\frac{d}{ds}u_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds}$:

$$2s = p_0 + 0 \qquad \Rightarrow \qquad p_0 = 2s$$

Substitution this into $p_0^2 + y_0 q_0 = u_0$ yields $q_0 = 2s - 4s^2$.

Combining the found relations gives

$$x = (4e^t - 3)s$$

$$y = e^t$$

$$u = (4e^{2t} - 3e^t)s^2$$

And finally,

$$u = \frac{yx^2}{4y - 3}$$

Problem 3. (6 points) *Solve the Cauchy problem*

$$x^2 u''_{xx} - 2x\, u''_{xy} + u''_{yy} + 2xu'_x - u'_y = 0.$$

a) *Determine the characteristic curves and type of equation*
b) *Transform the equation to the canonical form*
c) *Determine its general solution*

Solution.

(a) The principal symbol is $x^2 u''_{xx} - 2x\, u''_{xy} + u''_{yy}$, $a = x^2$, $b = -2x$ and $c = 1$. Since $b^2 - 4ac = 0$ the equation is parabolic. The characteristic line is found by integrating

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{1}{x}$$

That is: $y + \ln x = C$.

(b) Take the first coordinate $\lambda = y + \ln x$ and the second (since the equation is parabolic) we can choose arbitrarily, say, $\mu = y$. Then we find after change of the variables

$$x^2 u''_{xx} - 2x\, u''_{xy} + u''_{yy} + 2xu'_x - u'_y = u''_{\mu\mu} - u'_\mu = 0$$

(c) The solution of the latter equation is found as

$$u'_\mu = C_1(\lambda)e^\mu \quad \Rightarrow \quad u = C_1(\lambda)e^\mu + C_2(\lambda)$$

Hence the general solution in the old coordinates is

$$u = f(y + \ln x)e^y + g(y + \ln x)$$

for two arbitrary functions f, g.

Remark. This *form* of general solution is by no means unique, and there were several equivalent solutions found during the exam. For example, observe that

$$f(y + \ln x) = f(\ln(e^y x)) = F(e^y x)$$

Hence, for instance, $F(e^y x)e^y + G(e^y x) = \frac{F_1(e^y x)}{x} + G(e^y x)$ etc. will be equivalent forms of the general solution given above.

Problem 4. (6 points) *Solve the initial problem for the equation*

$$u''_{xx} - u''_{tt} = 2u'_t - 2u'_x, \qquad u(x,0) = xe^{-x}, \qquad u'_t(x,0) = 1.$$

Solution. Setting

$$v = e^{ax+bt}\, u$$

we find that for $a = b = 1$ the new function $v = ue^{x+t}$ is a solution to the wave equation $v''_{xx} - v''_{tt} = 0$. The new initial conditions are found then as

$$v(x,0) = e^x u(x,0) = x \equiv g(x)$$

$$v'_t(x,0) = e^x\big(u(x,0) + u'_t(x,0)\big) = x + e^x \equiv h(x)$$

By D'Alambert formula we find

$$v(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\, ds = x + xt + e^x \sinh t$$

and returning to the old function we find the required solution

$$u = e^{-x-t}(x + xt + e^x \sinh t)$$

Problem 5. (5 points) Let $U = \{(x,y)\colon\ 0 \le x \le \pi,\ \ 0 \le y \le \pi\}$. Solve the boundary value problem for the Laplace equation $u''_{xx} + u''_{yy} = 0$:

$$u(0,y) = 0, \qquad u(\pi,y) = \pi y, \qquad u(x,0) = 0, \qquad u(x,\pi) = 4\sin x + \pi x$$

Solution. Denote by u our solution and notice that xy is a harmonic function. Hence, the function $v(x,y) = u(x,y) - xy$ will be also harmonic with the new boundary conditions:

$$v(0,y) = v(x,0) = 0,$$

and

$$v(\pi,y) = \pi y - \pi y = 0, \quad v(x,\pi) = 4\sin x + \pi x - \pi x = 4\sin x$$

Thus we can apply the method of separating of variables: the solution is found as the sum

$$v(x,t) = A_1 \sin x \sinh y + A_2 \sin 2x \sinh 2y + \cdots$$

Comparing our boundary condition $v(x,\pi) = 4\sin x$ with the series we find $A_1 = \frac{4}{\sinh \pi}$ and $A_k = 0$ for $k \ge 2$. Hence $v = \frac{4}{\sinh \pi} \cdot \sin x \sinh y$. Returning to u we obtain finally:

$$u(x,y) = xy + \frac{4}{\sinh \pi} \cdot \sin x \sinh y \quad \blacksquare$$

Problem 6. (6 points) *Show that the characteristic function of the set* $A = \{(x,y) \in \mathbb{R}^2 : 2x + y > 0, y > 0\}$ *is a solution of the equation*

$$2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2} = c\,\delta_0$$

in the sense of distributions ($\delta_0(\phi) = \phi(0,0)$ is the Dirac delta-function). Find the constant c. (*Hint: the Gauss theorem* $\int_{\partial U} P dx + Q dy = \iint_U (Q'_x - P'_y) dxdy$).

Solution I (by using of the Gauss theorem). Denote the differential operator

$$Pu = 2\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}$$

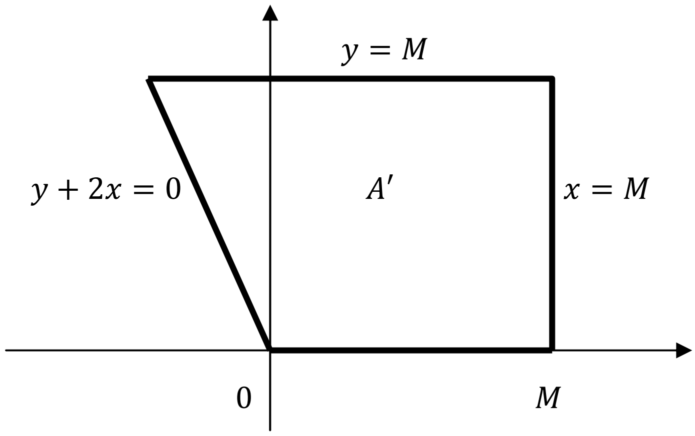
and by h denote the characteristic function of the set A**.** Notice that the adjoint operator $P^* = P$ because all the derivatives have even order. Then by the definition of weak derivative we have for any test function φ:

$$(Ph)\varphi = h\big(P^*(\varphi)\big) = \iint_{\mathbb{R}^2} h\, P\varphi\, dxdy = \iint_A \left(2\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial x^2}\right) dxdy$$

Since the support of φ is a compact set, we can choose $M > 0$ such that

$$\operatorname{supp} \varphi \subset [-M, M] \times [-M, M]$$

Then the integral above is taken over $A' = A \cap [-M, M] \times [-M, M]$ which is drown on picture below



Applying the Gauss theorem ($\int_{\partial U} P dx + Q dy = \iint_U (Q'_x - P'_y) dxdy$) to the latter integral with $Q = -\varphi'_x$ and $P = -2\varphi'_x$, we find

$$\iint_A \left(2\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial x^2}\right) dxdy = -\int_{\partial A} (2\varphi'_x dx + \varphi'_x dy) = -\int_{\partial A} \varphi'_x (2dx + dy)$$

The boundary integrals along the parts of lines $y = M$ and $x = M$ are equal to zero because $\varphi = 0$ there. The integral along the line $y + 2x = 0$ is also zero (since $2dx + dy = 0$). Finally we have only the integral along the horizontal part:

$$-\int_{\partial A} \varphi'_x(2dx+dy) = -\int_0^M 2\varphi'_x dx = -2\varphi(M,0) + 2\varphi(0,0) = 2\varphi(0,0).$$

This shows that $Ph = 2\delta_0$. Hence the required relation is proved and $c = 2$.■

Solution II (a direct proof). Starting as above, we obtain

$$(Ph)\varphi = \iint_A \left(2\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial x^2}\right) dxdy = 2\iint_A \frac{\partial^2\varphi}{\partial x\partial y} dxdy - \iint_A \frac{\partial^2\varphi}{\partial x^2} dxdy =: 2I_1 - I_2$$

We have for the first integral (since $\varphi'_y(M,y) = 0$)

$$I_1 = \iint_A \varphi''_{xy}\, dxdy = \int_0^M dy \int_{-\frac{y}{2}}^M \varphi''_{xy}(x,y)dx = \int_0^M \left(\varphi'_y(M,y) - \varphi'_y\left(-\frac{y}{2},y\right)\right) dy$$
$$= -\int_0^M \varphi'_y\left(-\frac{y}{2},y\right) dy$$

Obs!: the Newton-Leibniz formula is non-applicable to latter integral because

$$\varphi'_y\left(-\frac{y}{2},y\right) \neq \frac{d}{dy}\left(\varphi\left(-\frac{y}{2},y\right)\right) \equiv -\frac{1}{2}\varphi'_x\left(-\frac{y}{2},y\right) + \varphi'_y\left(-\frac{y}{2},y\right).$$

We find similarly

$$I_2 = \iint_A \varphi''_{xx}\, dxdy = \int_0^M dy \int_{-\frac{y}{2}}^M \varphi''_{xx}(x,y)dx = \int_0^M \left(\varphi'_x(M,y) - \varphi'_x\left(-\frac{y}{2},y\right)\right) dy$$
$$= -\int_0^M \varphi'_x\left(-\frac{y}{2},y\right) dy$$

Combing we obtain

$$2I_1 - I_2 = \int_0^M \left(-2\,\varphi'_y\left(-\frac{y}{2},y\right) + \varphi'_x\left(-\frac{y}{2},y\right)\right) dy = -2\int_0^M \frac{d}{dy}\left(\varphi\left(-\frac{y}{2},y\right)\right) dy$$
$$= -2\,\varphi\left(-\frac{M}{2},M\right) + 2\varphi(0,0) = 2\varphi(0,0)$$

Which yields again the required property.■

Problem 7. (6 points) *Let $u(x,t)$ be the weak solution to Burgers equation $uu'_x + u'_t = 0$ with the initial data*

$$u(x,0) = \begin{cases} 0, & x \le 0 \\ -x, & 0 \le x \le 1 \\ -1, & x \ge 1 \end{cases}$$

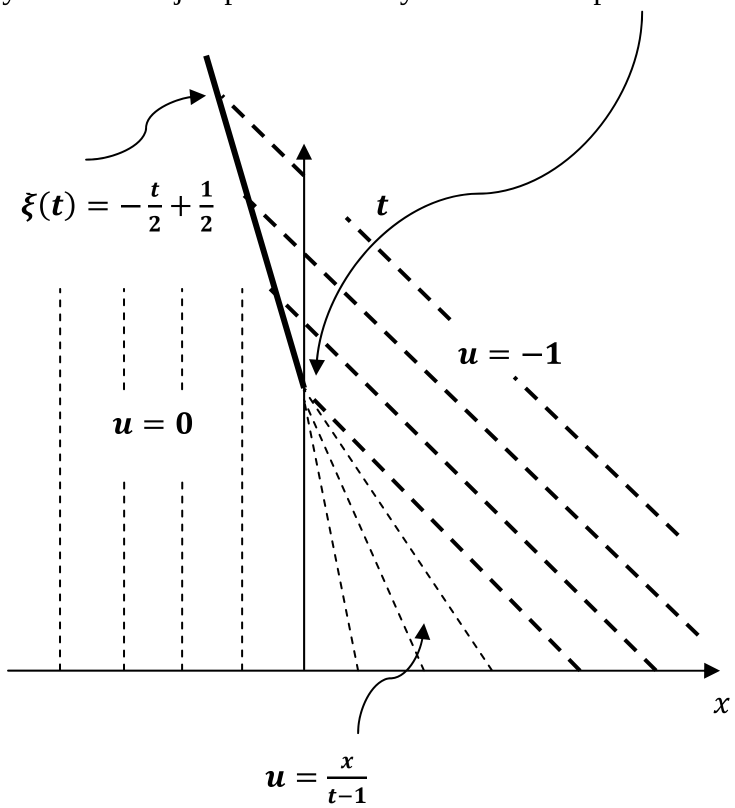
a) *By using the Rankine-Hugoniot condition find the equation of the shock line.*
b) *Find an exact form of the weak solution.*

c) *Also sketch the characteristics and shock curves (if any) in the (x, t)-plane.*

Solution. The characteristics (dashed lines)are given by

$$\begin{cases} x = x_0, & x_0 \leq 0 \\ x = -x_0 t + x_0, & 0 \leq x \leq 1 \\ x = -t + x_0, & x \geq 1 \end{cases}$$

Along these lines, before any crossings, the values of solution is given by $u = 0$, $u = \frac{x}{t-1}$ and $u = -1$ respectively. The initial jump discontinuity occurs at the point $x = 0, t = 1$



The Rankine-Hugoniot condition for the shock line gives

$$\xi'(t) = \frac{1}{2}(u_r + u_l) = -\frac{1}{2}$$

with $\xi(1) = 0$ (which corresponds to the discontinuity at $x = 0, t = 1$). Hence we find the shock line equation:

$$x = \xi(t) = -\frac{1}{2}t + \frac{1}{2}, t \geq 1$$

(displayed as the bold line on the picture).

Thus, we have found the following weak solution:

$$u(x, 0) = \begin{cases} 0, & x \leq 0, & t \leq 1 - \frac{x}{2} \\ \frac{x}{t-1}, & 0 \leq x \leq 1, & t < 1 - x \\ -1, & x \geq 1, & \text{otherwise} \end{cases}$$