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Course's contents:

- First-order equations (the method of characteristics, Cauchy problem for quasi-linear and fully non-linear equations, weak solutions).
- General techniques and principles for second-order equations
- The wave equation (d'Alembert formula, weak solutions, reflections, Duhamel's principle, two-dimensional and three-dimensional wave equation, conservation of energy).
- The Laplace equation (Mean Value Theorem and Maximum Principle, the fundamental solution, Green's function, Poisson kernel).
- The heat equation (Weak maximum principle, properties of the heat kernel, properties of the solution to the pure initial value problem).

Textbooks:

- Robert C. McOwen, Partial Differential Equations, Methods and Applications, Prentice Hall/Pearson Education, Inc., 2003 (Second Edition)

Optionally:

- * Lawrence C. Evans. Partial Differential Equations, AMS, Providence, RI. Series: Graduate Studies in Mathematics, Vol. 19, 1998.

Instruction: Lectures and problem solving sessions.

Examination: Written examination at the end of the course.

Some important examples of general PDE

Depending on the highest order of derivative involved, one speaks about PDE of first, second, third order etc.

Equation	Name	Order	Attributes
$u'_x = 0$	the simplest conservation law (transport equation)	1	the best equation
$u'_t + uu'_x = 0$	Inviscid Burger's equation	1	quasilinear
$u'_t + \operatorname{div} \mathbf{F}(u) = 0$	Scalar conservation law	1	quasilinear
$\Delta u \equiv u''_{xx} + u''_{yy} = 0$	Laplace Equation	2	linear homogeneous, *elliptic*
$\Delta u \equiv u''_{xx} + u''_{yy} = f(x, y)$	Poisson's equation	2	linear nonhomogeneous, *elliptic*
$u'_t - \Delta u = 0$	Heat (or diffusion) equation	2	linear homogeneous, *parabolic*
$u''_{tt} - \Delta u = 0$	The wave equation	2	linear homogeneous *hyperparabolic*,
$\operatorname{div} \dfrac{\nabla u}{\sqrt{1 + \lvert\nabla u\rvert^2}} = 0$	Minimal surface equation	2	quasilinear, *elliptic*
$u'_t + u'''_{xxx} + 6uu'_x = 0$	Korteweg–de Vries equation (KdV equation)	3	non-linear, integrable (solvable)

The 1st order PDE's in two variables:

For $u = u(x, y)$ one usually denotes the first derivatives by

$$p \coloneqq u_x \,, \qquad q \coloneqq u_y$$

so that the most general 1st order PDE is written as

$$F(x, y, u, p, q) \equiv F(x, y, u, u_x, u_y) = 0$$

Linear equations:

A **linear** equation (linear with respect to unknown function u)

$$a(x, y)\boldsymbol{u}'_x + b(x, y)\boldsymbol{u}'_y = c(x, y)$$

A linear equation may be **homogeneous** ($c \equiv 0$), e.g.

$$(x^2 - y^2)u'_x + 2xy\, u'_y = 0$$

or **nonhomogeneous,** e.g.

$$2u'_x + 3x\, u'_y = x + y$$

Quasilinear equations:

Generalizations of the linear case: an equation is called *quasilinear* if it is linear in first-partial derivatives of the unknown function u.

- a general **quasilinear** equation

$$a(x, y, \boldsymbol{u})u'_x + b(x, y, \boldsymbol{u})u'_y = c(x, y, \boldsymbol{u})$$

- a **semilinear** equation, if a and b independent of u:

$$a(x, y)u'_x + b(x, y)u'_y = c(x, y, \boldsymbol{u})$$

Fully non-linear equations:

$$F(x, y, u, p, q) = 0$$

with *F* chosen arbitrarily; then additionally required that

$$F_p'^2 + F_q'^2 \neq 0.$$

Example: $u_x^2 + u_y^2 = f(u)$ (that is the gradient of the unknown function is a function of the 'height').

An example of construction of a 1st order PDE

Wien's displacement law states that there is an inverse relationship between the wavelength λ of the peak of the emission of a black body and its temperature T:

$$T\lambda = b, \tag{1}$$

where $b = 2.897 \times 10^6$ nm $\cdot$ K is Wien's displacement constant.

This formulation is a typical example of a ***conservation law***:

- physically, a conservation law state that a certain physical property (energy, momentum etc) *does not change* in the course of time within an isolated physical system

- mathematically, this means that parameters of some system located on some *level-set* of the conserved quantity (function)

Most first-order partial differential equations are based on appropriate conservation laws and *vice versa* many 1st order PDE can be solved by determining the corresponding conservation law (called also the **first integrals**, or **general solution**).

Let us denote the left hand side in (1) by $u(T, \lambda)$. Then differentiating this by T and λ respectively, we find that

$$u_T = \lambda, \qquad u_\lambda = T,$$

that is we can write the following relation on the first derivatives:

$$\lambda u_T - T u_\lambda = 0. \tag{2}$$

Notice that the last equation has *many* solutions of the form

$$u(T, \lambda) \equiv T\lambda = C = const. \tag{3}$$

How to find some specific C?

- experimentally
- substitute some known data, say T_0 and λ_0, into the eq. (3)
- to use an additional conservation law, say $v(T, \lambda) = C_2$.

Remark. For an arbitrary function f, the function $u = f(T\lambda)$ is another solution to (2).

Motivation of the method of characteristics (homogeneous linear equation)

$$a(x,y)u'_x + b(x,y)u'_y = 0$$

A key idea is to interpret the equation as a conservation law for u itself by introducing some additional variable (time). Write ***characteristic equations***

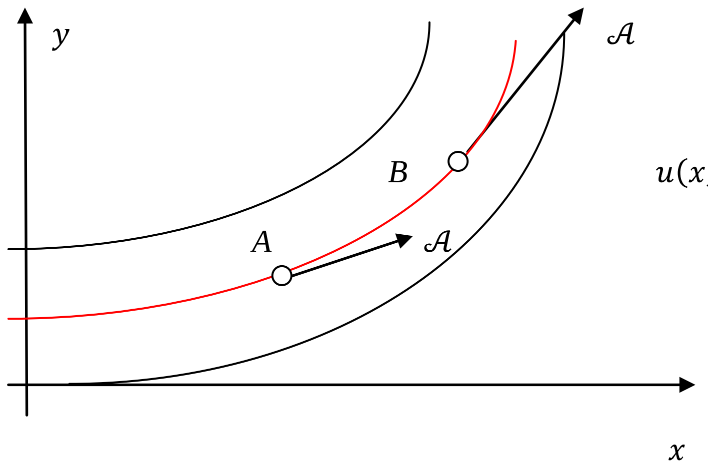
$$\frac{dx}{dt} = a(x,y), \qquad \frac{dy}{dt} = b(x,y)$$

If a and b 'good enough' (say, continuously differentiable in x and y), we can solve the system for any given initial conditions. Let $x(t)$ and $y(t)$ be such a solution. Then

$$\frac{du(x(t),y(t))}{dt} = u'_x \cdot \frac{dx}{dt} + u'_y \cdot \frac{dy}{dt} \equiv 0$$

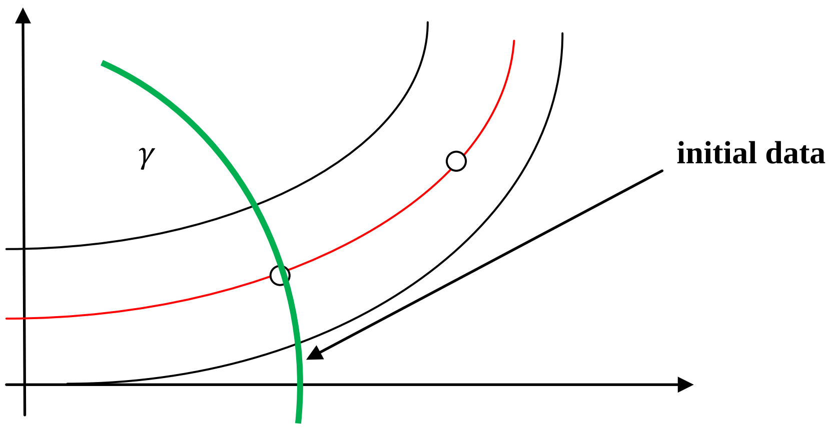
Remark. If $\mathcal{A} := \big(a(x,y),\ b(x,y)\big)$ is interpreted as a vector field in $\mathbb{R}^2$ then the above characteristic equations are *integral curves* for $\mathcal{A}$.

In particular, any solution u is constant along an integral curve:



$u(x,y) = \text{const}$ (**conservation property**)

In particular $u(A) = u(B)$ on the picture above and one can determine solution (*uniquely*) if one knows the values of the solution at some points. If one knows the values along a curve γ which is **transversal** to characteristic curves then one can find the solution by sweeping out an integral surface:



The method of characteristics for general quasilinear equation

$$a(x,y,u)u'_x + b(x,y,u)u'_y = c(x,y,u)$$

The Cauchy problem: given a curve Γ in $\mathbb{R}^3$, find a solution u of the first order equation whose graph contains Γ.

Cauchy data are usually given as a system of **parametric** equations, say

$$\Gamma: \quad x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s),$$

or **explicitly**, e.g.,

$$u(x,0) = h(x).$$

The curve Γ is always assumed **regular, that is**

$$x_0'^{\,2} + y_0'^{\,2} + u_0'^{\,2} \neq 0.$$

The method of characteristics amounts to solving the **characteristic equations**

$$\frac{dx}{dt} = a(x,y,u), \qquad \frac{dy}{dt} = b(x,y,u), \qquad \frac{du}{dt} = c(x,y,u). \tag{3}$$

Geometrically this is equivalent to finding integral surfaces. Consider a vector field

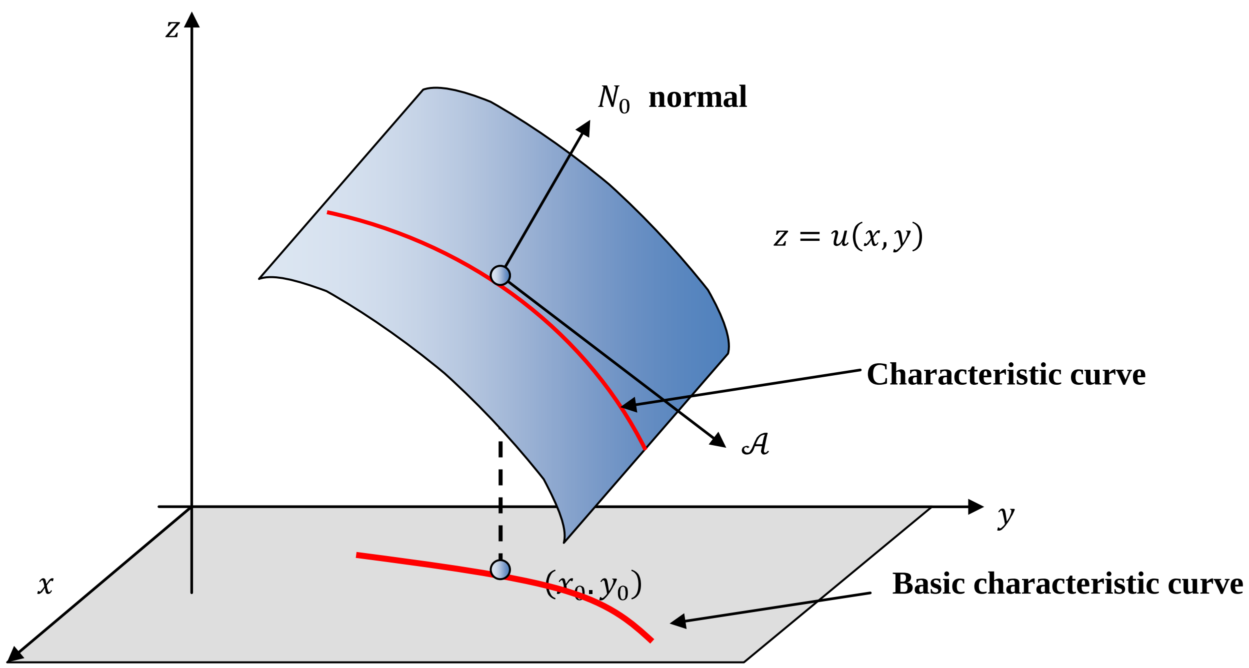
$$\mathcal{A} = \big(a(x,y,u), b(x,y,u), c(x,y,u)\big).$$

If $z = u(x,y)$ is the graph of some solution in $\mathbb{R}^3$ then geometrically equality

$$a(x,y,u)u'_x + b(x,y,u)u'_y - c(x,y,u) = 0$$

is equivalent to that $\mathcal{A}$ is *orthogonal* at $(x_0, y_0, u(x_0, y_0))$ to the **normal** vector

$$N_0 = (-u'_x(x_0.y_0), -u'_y(x_0.y_0), 1)$$



If the initial data Γ is *nowhere tangent* (= *transversal*) to the vector field $\mathcal{A}$ such a (regular) curve Γ is called **noncharacteristic**.

From linear algebra we find that Γ is noncharacteristic if and only if rank of the matrix

$$\operatorname{rank}\begin{pmatrix} x_0' & y_0' & z_0' \\ a|_\Gamma & b|_\Gamma & c|_\Gamma \end{pmatrix} = 2,$$

where $a|_\Gamma = a\big(x_0(s), y_0(s), u_0(s)\big)$ and so on.

Theorem. *If Γ is noncharacteristic, then the vector field $\mathcal{A}$ admits a unique integral (parametric)* ***surface*** *containing Γ. In particular, this yields the existence of a parametric solution for small values of parameter t.*

If, additionally, the determinant

$$\begin{vmatrix} x_0' & y_0' \\ a|_\Gamma & b|_\Gamma \end{vmatrix} \neq 0$$

there is an explicit solution $z = u(x, y)$ in a small neighborhood of the projection of Γ onto (x, y)-plane.

Sketch of the proof = Algorithm 1 (parametric method).

- Solve the characteristic system (3) as a system of ODEs with Cauchy data
$$x(0) = x_0(s), \qquad y(0) = y_0(s), \qquad u(0) = u_0(s),$$
where s is the parameter of parameterization of Γ.
- Thus we get a parametric parameterization of the solution for small enough t:
$$x = x(t, s), \qquad y = y(t, s), \qquad u = u(t, s).$$
- By the inverse function theorem this parametric function can be reduced locally to an explicitly given function $u = u(x, y)$ if the Jacobian
$$\begin{vmatrix} x_s' & y_s' \\ x_t' & y_t' \end{vmatrix} \neq 0$$
But for $t = 0$ the later determinant, by virtue of characteristic equations, coincides with
$$\begin{vmatrix} x_s' & y_s' \\ a|_\Gamma & b|_\Gamma \end{vmatrix}$$
which is non-zero. By continuity it is non-zero for small t.
- Elimination of s, t variables yields the required form $u = u(x, y)$.

■

Remark. The mean and principal technical difficulty is to solve the characteristic system.

Example 1. *Solve by method of characteristics* $u'_x + 2u'_y = 0$ *which graph passes through the curve* Γ *with parameterization* $x = s + s^2,\ y = 2s^2,\ u = s^2$

Solution. The characteristic equations are

$$\dot{x} = 1, \qquad \dot{y} = 2, \qquad \dot{u} = 0,$$

(the dot denotes the t-derivative). Hence we find

$$x = C_1 + t, \qquad y = C_2 + 2t, \qquad u = C_3\ .$$

We have

$$x(0) \equiv C_1 = s + s^2, \qquad y(0) \equiv C_2 = 2s^2$$

and

$$u|_\Gamma = C_3 = s^2.$$

Thus we obtain the following parameterization of our solution:

$$x = t + s + s^2, \qquad y = 2t + 2s^2, \qquad u = s^2.$$

This gives the ***parametric representation***.

In order to find explicit formulas, we eliminate the s, t variables as follows:

$$y = 2t + 2s^2 = 2t + 2u \qquad \Rightarrow \qquad t = \frac{y}{2} - u$$

and

$$x = t + s + s^2 = \left(\frac{y}{2} - u\right) + \sqrt{u} + u$$

This finally yields

$$u = \left(x - \frac{y}{2}\right)^2.$$