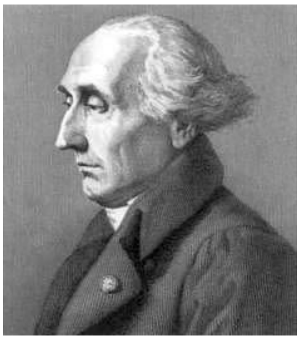
1. Lagrange method (non-parametric method of characteristics)

The fact that families of curves and surfaces can be defined by a differential equation means that the equation can be studied geometrically in terms of these curves and surfaces. The curves involved, known as *characteristic curves*, are very useful in determining whether it is or is not possible to find a surface containing a given curve and satisfying a given differential equation.

Leonhard Euler

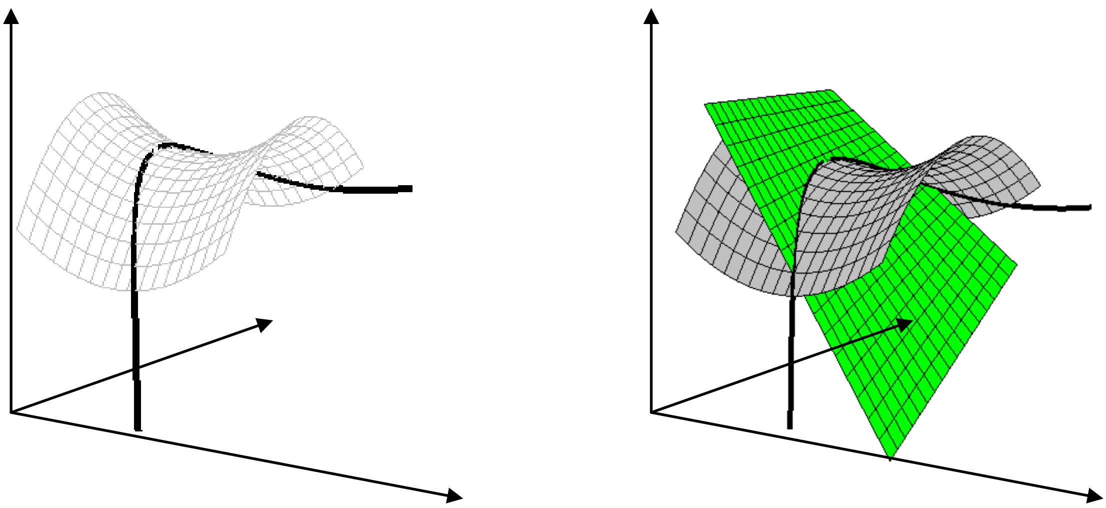
Gaspard Monge



Joseph-Louis Lagrange

This geometric approach to differential equations was begun by Leonhard Euler (1707-1783), Joseph-Louis Lagrange (1736–1813) and Gaspard Monge (1746–1818).

Lagrange method is based on the so-called *general solution* of a quasilinear equation. Instead of *parametric* representation of a characteristic curve (the left figure), one can define it *implicitly*, i.e. as an intersection of two surfaces (the right figure).



Parametric and implicit representation of a curve

Example 1. We illustrate the Lagrange method by the following equation

$$xu_x + yu_y = u.$$

Solution by the parametric method. The characteristic curves of the equation satisfy the system

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = y, \qquad \frac{du}{dt} = u \tag{1}$$

which has the following solution:

$$x = c_1 e^t, \qquad y = c_2 e^t, \qquad u = c_3 e^t.$$

If one knows some initial data then the solution is obtained as usual by substituting these data into the found relations.

Solution by the Lagrange method. If we are merely interested at the general solution, can eliminate t-variable as follows

$$\frac{u}{x} = \frac{c_3}{c_1}, \qquad \frac{y}{x} = \frac{c_2}{c_1}.$$

These relations can also be obtaining without recourse to the parametric method. Indeed, notice that the system of the characteristic equations (1) admits an equivalent *non-parametric* form (i.e. without the auxiliary variable t):

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

which yields

$$\frac{dx}{x} = \frac{dy}{y} \quad \Leftrightarrow \quad ydx - xdy = 0 \quad \Leftrightarrow \quad d\left(\frac{y}{x}\right) = 0 \quad \Leftrightarrow \quad \frac{y}{x} = C_1 = const,$$

and, similar,

$$\frac{dx}{x} = \frac{du}{u} \quad \Leftrightarrow \quad \frac{u}{x} = C_2 = const.$$

Analytically, this means that characteristic curves of our equation are located in intersection of two *integral surfaces*:

$$\phi := \frac{y}{x} = C_1, \qquad \psi := \frac{u}{x} = C_2.$$

Indeed, for $x \neq 0$ functions ϕ and ψ are *functionally independent*, that is their Jacobian has the maximal rank:

$$\operatorname{rank}\begin{pmatrix} \phi_x & \phi_y & \phi_u \\ \psi_x & \psi_y & \psi_u \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} & 0 \\ -\frac{u}{x^2} & 0 & \frac{1}{x} \end{pmatrix} = 2.$$

(equivalently, the gradients of ϕ and ψ are nowhere collinear). Thus, any solution $u = u(x, y)$ is defined implicitly as follows

$$f\left(\frac{y}{x}, \frac{u}{x}\right) = 0$$

for some choice of the function f. ■

Definition. By the *general solution* of a first-order, quasilinear partial differential equation

$$a(x, y, u)u_x' + b(x, y, u)u_y' = c(x, y, u)$$

we mean any relation of the form

$$f(\phi, \psi) = 0$$

where f is an arbitrary function, and ϕ and ψ are (independent) solutions to the *non-parametric* system

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}.$$

When $c \equiv 0$ the solution u is a function of ϕ, ψ. We illustrate this by the following example.

Example 2. Find the general solution of the linear equation

$$(y - z)u_x + (z - x)u_x + (x - y)u_x = 0.$$

Solution. The characteristic equations are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} = \frac{du}{0}.$$

Zero in the denominator of the last fraction means that u is constant along characteristics.

The remained part of the system is symmetric with respect to x, y and z, so we can find the symmetric combinations like $dx + dy + dz$ etc. To this end we denote by dt the common value of the above differentials so that

$$dx = (y - z)dt,$$

$$dy = (z - x)dt,$$

$$dz = (x - y)dt.$$

Summing the equations yields

$$dx + dy + dz = 0.$$

Then multiplying the first equation by x, the second by y, the third by z, and summing again we arrive at

$$xdx + ydy + zdz = 0.$$

These relations give two integrals

$$x + y + z = C_1, \qquad x^2 + y^2 + z^2 = C_2.$$

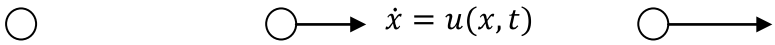
Intersection of these two surfaces determines a characteristic curve in $\mathbb{R}^3$. Since u must be constant along such a curve we get the general solution

$$u = f(x + y + z, x^2 + y^2 + z^2)$$

where f is an arbitrary function of three variables.

2. The inviscid Burgers' equation

Mechanical interpretation. 1D stream of particles is in motion, each particle having *constant velocity*; a velocity field is given by $u(x,t)$, where t denotes time.



If we follow an individual particle, we get a function $x = x(t)$, $t \geq 0$, for which velocity $u(x(t),t)$ remains constant. This yields

$$0 = \frac{d}{dt}\big(u(x(t),t)\big) = \text{the chain rule} = \; u\,u_x' + u_t'$$

or

$$u\,u_x' + u_t' = 0$$

Remark. The above equation has a specific form, $\left(\frac{u^2}{2}\right)_x' + u_t' = 0$, which is called a *conservation law* form. In general, any relation, like

$$A_x' + B_y' = 0$$

is a conservation law. For instance, $A_x' = 0$ expresses the fact that A has a constant value, while the general relation $A_x' + B_y' = 0$ can be interpreted as a *divergence free vector* field (A,B). Recall that such a vector field can always be written (at least locally) as a gradient of some function, called potential function of the vector field.

A complete form of the Burgers' equation contains an extra term (viscosity) and written as

$$u\,u_x' + u_t' = \varepsilon\, u_{xx}''.$$

It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow.

Johannes Martinus Burgers (1895-1981)

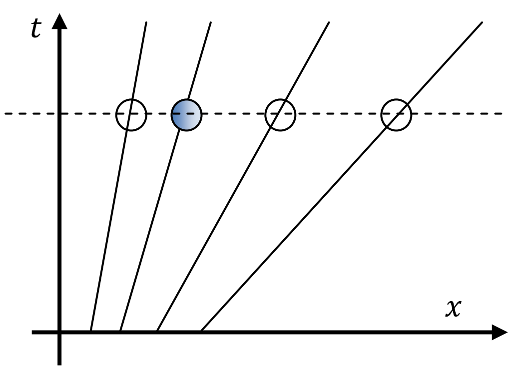
Now we return to the inviscid case and consider the initial problem velocity is given (to pose the Cauchy problem):

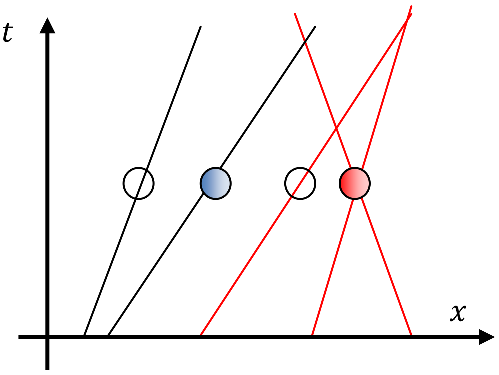
$$u(s,0) = h(s) \ = \text{ the velocity at the point } x = s \in \mathbb{R} \text{ and time } t = 0$$

- Characteristic system has the form
$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{0}$$
- Characteristics lines are **straight lines**:

$$z = h(s), \qquad x = h(s)\tau + s, \qquad t = \tau\,, \tag{2}$$

- In particular, u becomes constant along every characteristic
- The general solution (by the Lagrange method) is
$$u = h(x - ut)$$
- The characteristics can cross; moreover, if the initial condition is differentiable then the "*breaking*" *time* can be found explicitly:
$$T_{break} = -\frac{1}{\min\{h'(x)\}}$$
- If $h'(x) < 0$ (the left figure) the motion develops without collisions
- If $h'(x) < 0$ at some point then the solution will break and a *shock wave* will form.





(I) $h'(s) \geq 0$ for all s (II) $h'(s_0) < 0$ for some s_0

Indeed, a simple analysis of (2) shows that two characteristics will intersect if and only if the system

$$x = h(s_1)t + s_1, \qquad x = h(s_2)t + s_2$$

has a *positive* solution t_0 (i.e. for positive time), which is equivalent to

$$t_0 = -\frac{s_1 - s_2}{h(s_1) - h(s_2)} > 0$$

If this condition is satisfied for some values s_1 and $s_2$then solution suffers a *gradient catastrophe* type of singularity (otherwise solutions exist globally). An example of the gradient catastrophe:

Example 3. Consider an increasing initial data (regular traffic flow)

$$h(x) = \frac{e^x + 1}{e^x + 2}$$

Notice that $h(x) \sim \frac{1}{2}$ as $x \to -\infty$ and $h(x) \sim 1$ as $x \to +\infty$.

The picture below shows how velocity behaves as a function of coordinate x at three different 'times': $t = 0$ (green line), $t = 2$ (red line) and $t = 5$ (blue line). One can easily see how the profile functions move to the right: