Method of characteristic strips

Now we consider the general non-linear equation

$$F(x, y, u, p, q) = 0$$

where $u = u(x, y),\ p = u_x$ and $q = u_y$. We assume that F is regular in the sense that

$$F_p^2 + F_q^2 \neq 0.$$

The latter implies that locally either p or q can be found as a function of the remaining variables. In order to adjust the characteristics method to non-linear case one needs to "linearize" the initial non-linear equation. A key idea is to show that the first derivatives p and q satisfy *quasilinear* equations. To this end we differentiate $F = 0$ w.r.t. x and y (we replace u_{xx} by p_x, etc):

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0 \qquad (1)$$

$$F_y + F_u u_y + F_p p_y + F_q q_y = 0 \qquad (2)$$

But we have for the mixed partial derivatives:

$$p_y = u_{xy} = u_{yx} = q_x ,$$

hence equations (1) and (2) take the **quasilinear** form

$$F_p p_x + F_q p_y = -F_x - F_u p$$

$$F_p q_x + F_q q_y = -F_y - F_u q$$

Applying the characteristic equations to this system we get **four** equations

$$\frac{dx}{dt} = F_p, \qquad \frac{dy}{dt} = F_q, \qquad \frac{dp}{dt} = -F_x - F_u p, \qquad \frac{dq}{dt} = -F_y - F_u q$$

We need only one equation for u. Differentiating $u(x, y)$ w.r.t. t and applying the previous relations we get

$$\frac{du}{dt} = \frac{d}{dt} u(x, y) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = p \cdot F_p + q \cdot F_q$$

Thus we have arrived at the following system

$$\frac{dx}{dt} = F_p, \qquad \frac{dy}{dt} = F_q$$

$$\frac{du}{dt} = p \cdot F_p + q \cdot F_q$$

$$\frac{dp}{dt} = -F_x - F_u p, \qquad \frac{dq}{dt} = -F_y - F_u q$$

This system determines a family of integral curves in $\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$ and it is called the (*strips*) characteristic equations for the non-linear equation $F = 0$.

Remark. In general, in the n-dimensional, case one has a system of similar equations in $\mathbb{R}^{2n+1}$. In fact, let we have a 1st order non-linear equation

$$F(x, u, Du) = 0$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $Du = \left(u_{x_1}, \ \dots, u_{x_n}\right) = (p_1, \dots, p_n)$ is the gradient of $u = u(x_1, \dots, x_n)$. Then the modified system for characteristics is

$$\frac{dx_k}{dt} = F_{p_k}$$

$$\frac{dp_k}{dt} = -F_{x_k} - F_u p_k, \qquad k = 1, \dots, n$$

and

$$\frac{du}{dt} = \sum_{k=1}^{n} F_{p_k} p_k$$

Return to $n = 2$. We must complete our Cauchy conditions because we have now 5 ODE's but only 3 initial conditions. Since now we are in

$$\mathbb{R}^5 = \mathbb{R}^2_{xy} \times \mathbb{R}^2_{pq} \times \mathbb{R}^1_z$$

we need only the Cauchy data p_0 and q_0.

- We recall that the Cauchy condition can be written as a parameterized curve:

  $$\Gamma: \quad x = x_0(s), \;\; y = y_0(s), \;\; z = z_0(s)$$

  Substituting this into $F(x, y, z, p, q) = 0$ yields

  $$F(x_0, y_0, z_0, p_0, q_0) = 0 \tag{IC1}$$

- Another relation is found by differentiating the original initial condition (IC) with respect to the inner parameter s:

  $$\frac{d}{ds} z_0(s) = \frac{d}{ds} u\big(x_0(s), y_0(s)\big) = u_x'\big(x_0(s), y_0(s)\big) \cdot \frac{dx_0}{ds} + u_y'\big(x_0(s), y_0(s)\big) \cdot \frac{dy_0}{ds}$$

  This yields the so-called ***strip condition***:

  $$\frac{d}{ds} z_0(s) = p_0(s) \cdot \frac{dx_0}{ds} + q_0(s) \cdot \frac{dy_0}{ds} \tag{IC2}$$

These equations (IC1) - (IC2) provide two additional initial data, for p_0 and q_0.

Remark. In fact p_0 and q_0 need not to be uniquely defined and need not even exist. However, once p_0 and q_0 do exist, one can determine an integral surface

$$x = x(s,t), \;\; y = y(s,t), \;\; z = z(s,t)$$

which gives a parametric solution of the Cauchy problem for the non-linear equation $F = 0$.

Method of envelopes. In general, for the 1st order non-linear equation

$$F(x, y, u, p, q) = 0 \tag{*}$$

We are concerned with finding solutions u of (*) subject to the Cauchy condition

$$u = h \qquad \text{on } \Gamma.$$

Suppose that we have found a parametric family of general solutions, say

$$u = u(x, y; a, b).$$

Example 3. Consider equation $u_x = u_y^2$ with initial condition $u(0, y) = \frac{y^2}{2}$. Consider the function

$$u = a + bx + cy + dxy.$$

Then for $d = 0$ and $b = c^2$ this function solves our equation (the parameter a can be chosen arbitrarily). This yields a parametric family of solutions (after changing notation $b \leftrightarrow c$):

$$u = a + b^2 x + by.$$

Let us introduce the matrix

$$J(u) \coloneqq \begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix}$$

Definition: A function $u = u(x, y; a, b)$ of class C^2 is called a *complete integral* provided there are some open subsets U and V such that

(i) $u(x, y; a, b)$ is a solution of equation (*) for any $(x, y) \in U$ and $(a, b) \in V$

and

(ii) $\operatorname{rank} J(u) = 2$ for any $(x, y) \in U$ and $(a, b) \in V$.

Theorem. *Let $u(x, y; a, b)$ be a complete integral for $F = 0$. Suppose that the system*

$$\partial_a u = \partial_b u = 0 \tag{**}$$

has a solution $a = \varphi(x, y), b = \psi(x, y)$. Then the ***envelope function*** *$v(x) = u(x, y; \; \varphi, \psi)$ solves also the original equation $F = 0$.*

Idea of the proof: We have

$$\partial_x v(x,y) = u_x(x,y;\phi,\psi) + u_a(x,y;\phi,\psi)\cdot\phi_x + u_b(x,y;\phi,\psi)\cdot\psi_x$$

where $u_a = u_b = 0$ for $a = \varphi(x,y)$ and $b = \psi(x,y)$ by our assumption (**).

Hence

$$\partial_x v(x,y) = u_x(x,y;\phi,\psi)$$

and it easily follows that the envelope function satisfies also $F = 0$. ■

How it works?

In practice one usually uses a one parametric envelope solution which can be found by substituting some auxiliary function $b = B(a)$ or $a = A(b)$ in $u(x,y;a,b)$. We demonstrate this below.

Example 3 (cont). We see that the our Jacobian matrix has the maximal rank:

$$\operatorname{rank}\begin{pmatrix} u'_a & u''_{ax} & u''_{ya} \\ u'_b & u''_{bx} & u''_{yb} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 \\ 2bx+y & 2b & 1 \end{pmatrix} = 2$$

Hence we are in position of the Theorem.

(i) Set $a = kb^2$, where the constant k will be chosen later. We have

$$v = kb^2 + b^2x + by$$

and the **envelope equation** is

$$0 = \frac{\partial}{\partial b} v = 2kb + 2bx + y,$$

hence

$$b = -\frac{y}{2x+2k}$$

(ii) Substituting this into v we find

$$v(x,y;a,b) = -\frac{y^2}{4(x+k)}$$

(iii) Finally applying our Cauchy condition we find $k = -\frac{1}{2}$. Hence the desired solution is

$$u(x,y) = \frac{y^2}{2-4x}.$$

Question: Why $a = kb^2$? Check that the above argument breaks down for $a = kb$