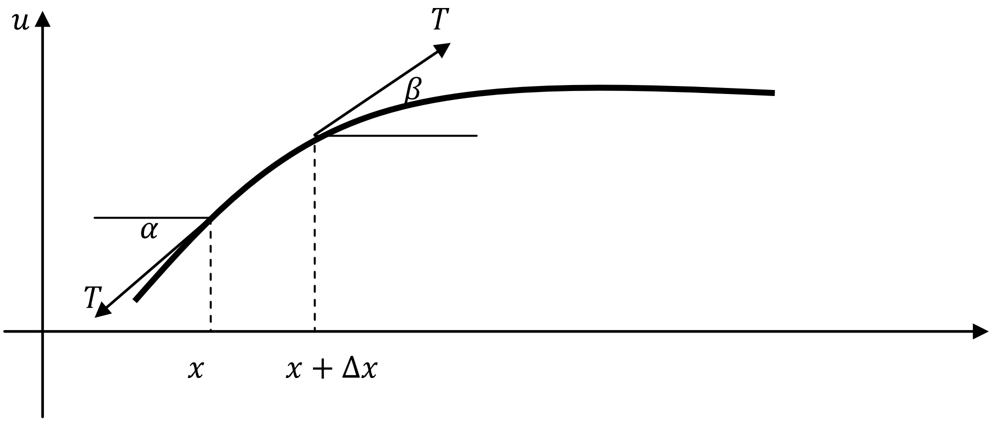
Lecture 4: Introduction in higher order PDE

Mathematical models: derivation of the wave equation

Let us consider a stretched string of a finite length and fixed at the end points. The problem is to determine the equation of motion which characterizes the position $u(x,t)$ of the string at time t after an initial disturbance is given.



In order to obtain a simple equation, we make the following main assumptions:

1. The string is flexible and elastic, the tension in the string is always in the direction of the tangent to the existing profile of the string and the tension is constant.
2. The deflection is small compared with the length of the string and the slope of the displaced string at any point is small compared with the length of the string.

Denote by T the tension at the end points as shown in the figure above. By Newton's second law of motion, the forces acting on the element of the string in the vertical direction are equal to the element's mass times the acceleration:

$$T\sin\beta - T\sin\alpha = \Delta m \cdot u_{tt} \approx \rho \cdot \Delta x \cdot u_{tt}\ .$$

(by our assumption $\sqrt{(\Delta u)^2 + (\Delta x)^2} \approx \Delta x$). On the other hand, since the angles α and β are small enough, we also have $\sin\alpha \approx \tan\alpha$, $\sin\beta \approx \tan\beta$, thus

$$\tan\beta - \tan\alpha = \frac{\rho\Delta x}{T} \cdot u_{tt}.$$

But we know that $\tan\alpha = u_x(x,t)$ and $\tan\beta = u_x(x+\Delta x, t)$, hence

$$\tan\beta - \tan\alpha = u_x(x+\Delta x, t) - u_x(x,t) \approx u_{xx}(x,t)\ \cdot \Delta x,$$

and finally we obtain the one-dimensional *wave equation*:

$$u_{xx} - \frac{\rho}{T}u_{tt} = 0.$$

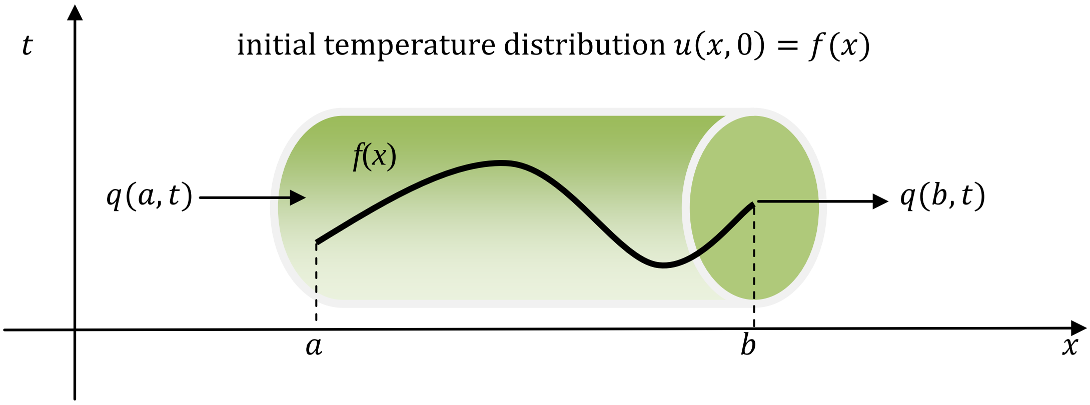
The initial disturbance may be interpreted then as the initial conditions:

$$u(x,0) = f(x), \qquad u_t(x,0) = g(t),$$

where $f(x)$ and $g(x)$ describe the initial profile and the initial velocity of the string respectively.

Conservation laws and derivation of the heat equation

We consider the flow of heat along a metal *insolated* rod



Let $u = u(x, t)$ be temperature at time t at a given point x and A be the cross sectional area of the rod. Then *energy* of an arbitrary piece of rod from a to b is

$$E = \int_a^b u(x, t) \cdot c \cdot \underbrace{\rho \; A \; dx}_{dm = \text{mass}}$$

Here c is the specific heat capacity of the rod.

The wave heat flow:

$$R = A\big(q(a, t) - q(b, t)\big) = -A \int_a^b \frac{\partial}{\partial x} q(x, t)\, dx$$

Conservation of energy (in terms of power = time-derivative of energy):

$$R = \frac{\partial E}{\partial t}$$

implies the *integral form* of the heat equation:

$$\int_a^b \left(\rho \; c \cdot \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} q(x, t) \right) dx = 0.$$

By virtue of arbitrariness of a and b we find

$$\rho \; c \cdot \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} q(x, t) = 0.$$

Finally, by using the Fourier law $q(x, t) = -\lambda \frac{\partial}{\partial x} q(x, t)$ we arrive at (the differential form of) the heat equation:

$$\frac{\partial u}{\partial t} - C \frac{\partial^2 u}{\partial x^2} = 0.$$

Analytic theory: the normal form of a higher order PDE

Multi-index notation

- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_k \in \mathbb{N}$
- $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\qquad \alpha! = \alpha_1!\,\alpha_2! \dots \alpha_n!$

Products and derivative:

- $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
- $x^\alpha = x^{\alpha_1} x^{\alpha_2} \dots x^n$,
- $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$

Commutative relations:

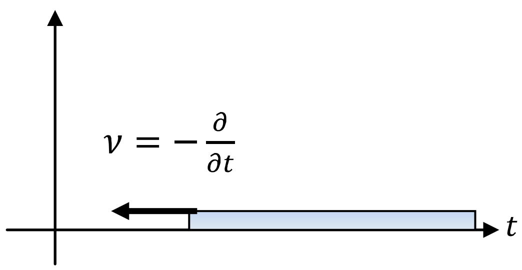
$$D^\alpha D^\beta = D^\beta D^\alpha = D^{\alpha+\beta}$$

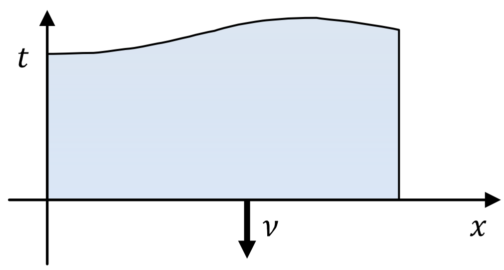
Definition: A general m-order PDE is an equation of the kind

$$F(x, D^\alpha u) = 0\ ,\ |\alpha| \le m \qquad (*)$$

Cauchy problem:

- For the 1st order PDE with 2 variables we consider Cauchy $u|_\gamma$ data on a curve γ in $\mathbb{R}^2$ or on a surface for 3 variables, etc.
- It is natural to assume that Cauchy problem for (*) involves replacing the initial curve γ in $\mathbb{R}^2$ by an initial $(n-1)$-dimensional surface S in $\mathbb{R}^n$, but we still need to involve some additional data, say the normal derivatives of u along S.





$\nu = -\frac{\partial}{\partial t}$ the outward normal

- What does it mean that S is *noncharacteristic* in this new set-up?

Definition: By the Cauchy data for the m-th order PDE (*) we understand the set of values

$$u,\ \frac{\partial u}{\partial \nu},\ \frac{\partial^2 u}{\partial \nu^2},\ \dots,\ \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \qquad (**)$$

along a ***hypersurface*** S, where ν is the unit *normal vector* to S. Moreover, the surface S is called ***noncharacteristic*** for (*) if the derivatives (**) on S determine all derivatives of the solution u on S.

Example 1. Consider the Cauchy problem in the upper half-plane $y \geq 0$:

$$u_{xx} + uu_{yy} - 2u = 0, \qquad u(x,0) = 1, \qquad u_y(x,0) = x.$$

Notice that the unit (outward) normal to the boundary of the half-plane is vector $\nu = (0, -1)$ which corresponds to the derivative ∂_y. Write our equation as follows

$$u_{yy} = 2 - \frac{1}{u} u_{xx}.$$

Then we know already $u(x,0) = 2$, $u_y(x,0) = x$ and in order to find the second derivative

$$u_{yy}(x,0) = 2 - \frac{1}{u(x,0)} u_{xx}(x,0) = 2 - u_{xx}(x,0)$$

we need $u_{xx}(x,0)$. This derivative can be found by differentiating the initial condition $u(x,0) = 1$ two times:

$$u_{xx}(x,0) = \frac{\partial^2}{\partial x^2}\big(u(x,0)\big) = 0.$$

Hence $u_{yy}(x,0) = 2$. We can also find the mixed dderivative

$$u_{xy}(x,0) = \frac{\partial}{\partial x}\Big(u_y(x,0)\Big) = 1.$$

Thus we already know all first and second derivatives. Similarly we find

$$u_{xxx}(x,0) = \frac{\partial^3}{\partial x^3}\big(u(x,0)\big) = 0,$$

$$u_{xxy}(x,0) = \frac{\partial^2}{\partial x^2}\Big(u_y(x,0)\Big) = 0,$$

$$u_{xyy}(x,0) = \frac{\partial^1}{\partial x^1}\Big(u_{yy}(x,0)\Big) = 0,$$

and the last third derivative is found by differentiating the origin equation:

$$u_{yyy} = \frac{\partial}{\partial y}\Big(2 - \frac{1}{u} u_{xx}\Big) = \frac{1}{u^2} u_{xx} - \frac{1}{u} u_{xxy},$$

hence we have

$$u_{yyy}(x,0) = \frac{1}{u^2(x,0)} u_{xx}(x,0) - \frac{1}{u(x,0)} u_{xxy}(x,0) = 0.$$

Thus we obtain all higher derivatives by the induction procedure. In our case we see moreover that all derivatives starting with order 3 are zero. In particular, u is a polynomial:

$$u(x,y) = u(0,0) + u_x(0,0)x + u_y(0,0)y + u_{xy}(0,0)xy + \frac{u_{yy}(0,0)y^2}{2} = 1 + xy + y^2. \ \blacksquare$$

The above procedure can be generalized as follows. Namely, thus defined noncharacteristic property will be true, if for example, we can express the initial equation in the form

$$\frac{\partial^m u}{\partial \nu^m} = G(x, D^\alpha u)$$

where the R.H.S. does not contain $\frac{\partial^m u}{\partial \nu^m}$.

Definition. The above is called the *normal form* of (*) with respect to the hypersurface *S.*

By local changing of the coordinates $(x_1, x_2, \ldots, x_n)$, one can assume that S is 'straightened out', i.e. it coincides with a hyperplane in new coordinates.

Example 2. Consider a surface S given by $x_3 = x_1 x_2 + 1$. Then in the new coordinates

$$\tilde{x}_1 = x_1, \quad \tilde{x}_2 = x_2, \quad \tilde{x}_3 = x_3 - x_1 x_2 - 1$$

the surface is given as $\tilde{x}_3 = 0$. It is easy to show that the Jacobian $\frac{\partial(\tilde{x}_1, \tilde{x}_1, \tilde{x}_1)}{\partial(x_1, x_2, x_3)} = 1 \neq 0$. ■

Thus, without loss of generality we can assume that S is given by $x_n = 0$ in some coordinates $(x_1, \ldots, x_n)$ and therefore the normal derivatives coincide with the corresponding derivative w.r.t. the distinguished coordinate x_n:

$$\frac{\partial^k u}{\partial \nu^k} = \frac{\partial^k u}{\partial x_n^k}$$

Hence our Cauchy problem is rewritten in this "flat" case as follows:

$$\left.\frac{\partial^k u}{\partial x_n^k}\right|_{x_n=0} = g_k(x_1, \ldots, x_{m-1}), \qquad \forall k = 0, 1, 2, \ldots, m-1$$

In general (as in Example 1), for en equation given in the normal form, the Cauchy data determines *all* derivatives of the solution u on S. Indeed, the derivatives can be found step by step, by differentiating the initial conditions and the normal equation.

Augustin Louis Cauchy (1789-1857) Sofia Vasilyevna Kovalevskaya (1850-1891)

A deeper result is the celebrated

Cauchy-Kovalevski Theorem: *If all* g_k, $k = 0,1, \dots, m-1$ *are real* analytic *in a neighborhood of* $0 \in \mathbb{R}^{n-1}$ *, and if G is real analytic in a neighborhood of* $(0, D^{\alpha}u(0))$*, then there exists a unique real analytic solution* u *of the above Cauchy problem in some neighborhood of* $0 \in \mathbb{R}^n$*.*

Shortcomings of Cauchy-Kovalevski Theorem:

- The theorem is non-effective in practical questions
- It fails to recognize well posed non-analytic Cauchy problems

What is a well posed problem?

In physics, one expects a stability of solutions with respect to their initial conditions, because a small change of data should induce only a small change in the solution. Otherwise, the solution becomes meaningless.

Definition: A problem is *well posed* (in the sense of Hadamard) if a solution exists, is unique, and depends continuously on its data.

Lewy example: there exists a complex-valued function $F(x,y) \in C^{\infty}$ such that the differential equation

$$u'_x + ixu'_y = F(x,y)$$

has no solutions at all. Hence, the analog of the Cauchy-Kovalevskaya theorem fails for equations with the smooth (infinitely many times differentiable) coefficients.

Example 3 (McOwen, p.48). Find solution of the Cauchy problem

$$u_{yy} = u_{xx} + u, \qquad u(x,0) = e^x, \;\; u_y(x,0) = 0,$$

in the form of power series expansion $\sum_{n=1}^{\infty} a_n(x) y^n$.

Solution. In assumption that our solution is an analytic function we have

$$a_n(x) = \frac{1}{n!} \partial_y^n u(x,y)|_{y=0}.$$

We have easily $a_0 = e^x$ and $a_1 = 0$. Next,

$$\partial_y^2 u(x,0) = u_{xx}(x,0) + u(x,0) = \frac{\partial^2}{\partial x^2}\big(u(x,0)\big) + e^x = 2e^x \quad \Rightarrow \quad a_2 = e^x.$$

And by induction we find that $\partial_y^{2n} u(x,0) = 2^n e^x$, that is

$$a_{2n} = \frac{2^n}{(2n)!}\, e^x$$

For a_3 we have: $3!\, a_3 = u_{yyy}(x,0) = u_{xxy}(x,0) + u_y(x,0) = 0$. By induction one can prove that $a_{2n+1} = 0$ for all $n \geq 0$. Thus we have

$$u(x,y) = \sum_{n=1}^{\infty} a_n(x) y^n = e^x \sum_{n=1}^{\infty} \frac{2^n y^{2n}}{(2n)!} = e^x \cosh \sqrt{2}\, y.$$