# Lecture 5. Classification of the second-order equations in two variables

## Characteristics for second-order equations

The general *linear* second-order partial differential equation in two variables is

$$au_{xx} + bu_{xy} + cu_{yy} + d_1u_x + d_2u_y + eu + f = 0 \qquad (1)$$

where $a, b, c, \ldots$ are some twice continuously differentiable functions of $x, y$. Then

$$au_{xx} + bu_{xy} + cu_{yy}$$

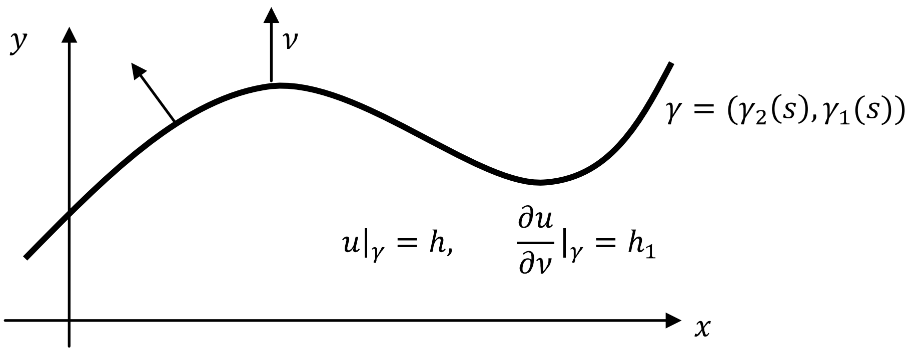
is called sometimes the *principal part* of (1). The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry. Recall that the equation

$$A\xi^2 + B\xi\eta + C\eta^2 + D_1\xi + D_2\eta + E = 0$$

represents *hyperbola*, *parabola*, or *ellipse* accordingly as $B^2 - 4AC$ is positive, zero, or negative. The classification of second-order equations is based upon the possibility of reducing equation (1) by coordinate transformation to canonical or standard form at a point.

We start with the following problem: given curve $\gamma \subset \mathbb{R}^2\ (x, y)$, determine when $\gamma$ is a *characteristic* curve, i.e. when does the Cauchy data along $\gamma$ *not determine* all derivatives of the solution along $\gamma$.

Consider the equation (1) above with the Cauchy data along $\gamma$:




As usual, $\nu$ is the unit *normal vector* to $\gamma$. Writing $\nu = (\nu_1, \nu_2)$ we obtain

$$\frac{\partial u}{\partial \nu} = \nu_1 u_x + \nu_2 u_y,$$

hence our Cauchy problem is equivalent to the following

$$u|_\gamma = h, \qquad \frac{\partial u}{\partial x}|_\gamma = \varphi, \qquad \frac{\partial u}{\partial y}|_\gamma = \psi.$$

Notice that thus transformed Cauchy data contains already *three* equations, that is, these equations are dependent. An additional condition is the so-called *compatibility* equation

$$h'(s) = \varphi(s)\gamma_1'(s) + \psi(s)\gamma_2'(s).$$

Here $\big(\gamma_2(s), \gamma_1(s)\big)$ is a parameterization of $\gamma$. Indeed, the compatibility equation follows from the equality $u\big(\gamma_1(s), \gamma_2(s)\big) = h(s)$ by taking derivative with respect to $s$:

$$h'(s) = \frac{d}{dx} u\big(\gamma_1(s), \gamma_2(s)\big) = u_x\big(\gamma_1(s), \gamma_2(s)\big)\gamma_1'(s) + u_y\big(\gamma_1(s), \gamma_2(s)\big)\gamma_2'(s).$$

Furthermore, taking the second derivatives (along $\gamma$) we obtain:

$$\varphi'(s) \equiv (u_x')_s' = u_{xx}''\gamma_1{}' + u_{xy}''\gamma_2'(s)$$

$$\psi'(s) \equiv \big(u_y'\big)_s' = u_{xy}''\gamma_1'(s) + u_{yy}''\gamma_2'(s)$$

Now combining these equations with (1) we obtain a linear system on the *second derivatives*:

$$\begin{aligned} \gamma_1' u_{xx}'' + \gamma_2' u_{xy}'' \qquad\qquad &= \varphi'(s) \\ \gamma_1' u_{xy}'' + \gamma_2' u_{yy}'' &= \psi'(s) \\ a u_{xx} + b u_{xy} + \; c u_{yy} &= Q \equiv -(d_1 u_x + d_2 u_y + eu + f) \end{aligned}$$

which is *uniquely solvable* provided that the determinant

$$D = \begin{vmatrix} \gamma_1' & \gamma_2' & 0 \\ 0 & \gamma_1' & \gamma_2' \\ a & b & c \end{vmatrix} = a{\gamma_2'}^2 - b\gamma_1'\gamma_2' + c\,{\gamma_1'}^2 \neq 0$$

In particular, the Cauchy data determines *all second-order derivatives* along $\gamma$ if and only if $D \neq 0$. It is natural, therefore, to call $\gamma$ to be characteristic if

$$a{\gamma_2'}^2 - b\gamma_1'\gamma_2' + c\,{\gamma_1'}^2 = 0.$$

**<u>The principal symbol</u>**

Let us introduce the following quadratic form:

$$\sigma(\xi_1, \xi_2; x, y) = \; a(x, y)\xi_1^2 + b(x, y)\xi_1\xi_2 + c(x, y)\xi_2^2$$

One distinguishes the following three cases (depending on the point $(x, y)$:

(i) $b^2 - 4ac > 0$, there are two characteristics, and (1) is called *hyperbolic*
(ii) $b^2 - 4ac = 0$, there are only one characteristic, and (1) is called *parabolic*
(iii) $b^2 - 4ac < 0$, there are no characteristics, and (1) is called *elliptic*

**Examples:**

- The wave equation $u''_{xx} - u''_{yy} = 0$ is *a hyperbolic equation*
- The heat equation $u''_{xx} - u'_{y} = 0$ is *a parabolic equation*
- The Laplace equation $u''_{xx} + u''_{yy} = 0$ is *an elliptic equation*

**How to find characteristics?**

If $\gamma$ is given as a graph $y = y(x)$ then

$$\gamma_1 = s, \qquad \gamma_2 = y(s)$$

hence $D = 0$ becomes

$$ay'^2 - by' + c = 0.$$

This proves that characteristics satisfy an ordinary differential equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

**How to reduce a principal linear PDE to a canonic form?**

First solve the above ODE to determine exact form of solutions $y_1(x)$ and $y_2(x)$. In the hyperbolic case one introduces the new coordinates, say,

$$\lambda(x, y) = y_1(x) - y \quad \text{and} \quad \mu(x, y) = y_2(x) - y$$

which diagonalize the principal part. Next, one calculates the old derivatives $u'_x,\ u''_{xx}$ etc. in the new coordinates.

**Example (1998-03-16)**

*Study the equation*

$$\frac{3}{4}u''_{xx} - 2y\, u''_{xy} + y^2 u''_{yy} + \frac{1}{2}u'_x = 0$$

a) *Where the equation is hyperbolic?*
b) *Determine the characteristic curves.*
c) *Transform the equation to canonical form where this is possible.*
d) *Determine the general solution in the domain where it is hyperbolic.*

<u>Solution.</u>

a) The principal symbol is $\frac{3}{4}\xi_1^2 - 2y\, \xi_1\xi_2 + y^2\xi_2^2$ with discriminant

$$b^2 - 4ac = 4y^2 - 3y^2 = y^2 \geq 0.$$

Hence our equation is *hyperbolic* everywhere outside the $x$-axis, that is when $y \neq 0$, where our equation has parabolic type. Denote by $U = \{(x, y): y \neq 0\}$.

b) The characteristic equation has the form:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2y \pm y}{3/2} = \{-2y, -\frac{2}{3}y\}$$

We assume that $y$ >0 (the remained case is symmetric). Then by solving $y'(x) = -2y$ and $y'(x) = -\frac{2}{3}y$ we find equations for the characteristic curves:

$$\lambda := \ln y + 2x = const, \qquad \mu := 3\ln y + 2x = const$$

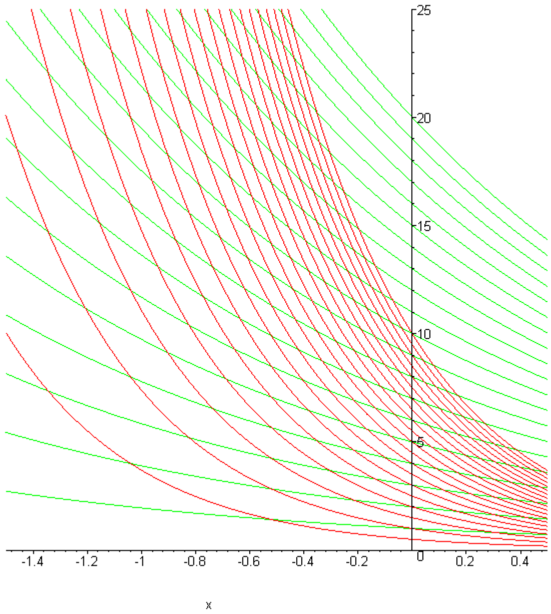



**Figure 1** The characteristic lines: $\boldsymbol{\mu = const}$ colored by green and $\boldsymbol{\lambda = const}$ colored by green

c) We have $y = E^{-1}$, where $E = \exp\frac{\lambda-\mu}{2}$ and $x = \frac{3\lambda-\mu}{4}$. In the new coordinates

$$u'_x = 2u'_\lambda + 2u'_\mu$$

$$u'_y = \frac{1}{y} \cdot u'_\lambda + \frac{3}{y} \cdot u'_\mu = (u'_\lambda + 3\,u'_\mu) \cdot \exp\frac{\lambda-\mu}{2}$$

and setting , we find

$$u''_{xx} = 2 \cdot (u'_x)'_\lambda + 2 \cdot (u'_x)'_\mu = 4u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 4u''_{\mu\mu}$$

$$u''_{xy} = 2 \cdot [(u'_\lambda + 3\,u'_\mu)E]'_\lambda + 2 \cdot [(u'_\lambda + 3\,u'_\mu)E]'_\mu = (2u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 6u''_{\mu\mu} + 2u'_\mu)E$$

$$u''_{yy} = [(u'_\lambda + 3\,u'_\mu)E]'_\lambda \cdot E + 3 \cdot [(u'_\lambda + 3\,u'_\mu)E]'_\mu \cdot E = (4u''_{\lambda\lambda} + 8u''_{\lambda\mu} + 4u''_{\mu\mu})E^2$$

Hence substituting the found relations into initial equation yields

$$\frac{3}{4}u''_{xx} - 2y\, u''_{xy} + y^2 u''_{yy} + \frac{1}{2}u'_x = -4u''_{\lambda\mu} - 4u'_\mu = 0$$

We have

$$u''_{\lambda\mu} + u'_\mu = 0$$

is the desired canonical form.

d) In order to find the general solution we write the latter equation as $(u'_\lambda + u)'_\mu = 0$, hence

$$u'_\lambda + u = f(\lambda)$$

for arbitrary $f(\lambda)$. Solving this linear ODE yields

$$u = g(\mu)e^{-\lambda} + f_1(\lambda)$$

for a new function $f_1$. Thus the general solution is found as

$$u = F_1(\mu)e^{-\lambda} + F_2(\lambda)$$

where both function can be chosen arbitrarily. ■