Lecture 6: The one-dimensional homogeneous wave equation

We shall consider the one-dimensional homogeneous wave equation for an infinite string

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in \mathbb{R}, \;\; t > 0.$$

Recall that the wave equation is a hyperbolic 2nd order PDE which describes the propagation of waves with a constant speed $c \neq 0, \;\; c \in \mathbb{R}$.

By the method of characteristics described earlier, the characteristic equation according to equation for the wave equation is

$$dx^2 - c^2 dt^2 = 0$$

which reduces to $dx \pm c dt = 0$. The integrals are straight lines

$$x + ct = c_1, \qquad x - ct = c_2,$$

Introducing the characteristic coordinates

$$\lambda := x + ct, \qquad\qquad \mu := x - ct$$

we find the canonical form

$$-4c^2\, u_{\lambda\mu} = 0$$

which implies (because $c \neq 0$)

$$u_{\lambda\mu} = 0.$$

Integrating with respect to λ, we obtain

$$u_\mu = g(\mu),$$

where g is an arbitrary function of μ. Integrating again with respect to μ, we find

$$u = \int g(\mu) d\,\mu + F(\lambda) = G(\mu) + F(\lambda),$$

where F and G are arbitrary twice differentiable functions. The *general solution* of the latter equation is then found as

$$u(x,t) = F(x + ct) + G(x - ct). \qquad (*)$$

Remark. The fact that $u(x,t)$ is a sum of two functions in one variable usually is interpreted as a superposition of two waves propagating with a constant shape in *opposite* directions along the x-axis.

As an example we consider the superposition of two (soliton-like) waves with generating functions (amplitudes)

$$F(x) = \frac{1}{\cosh(x-1)}, \qquad G(x) = \frac{4}{\cosh(x+2)}$$

Waves with different times are pictured by different colors (see Figure 1) and the full time evolution is shown on Figure 2:

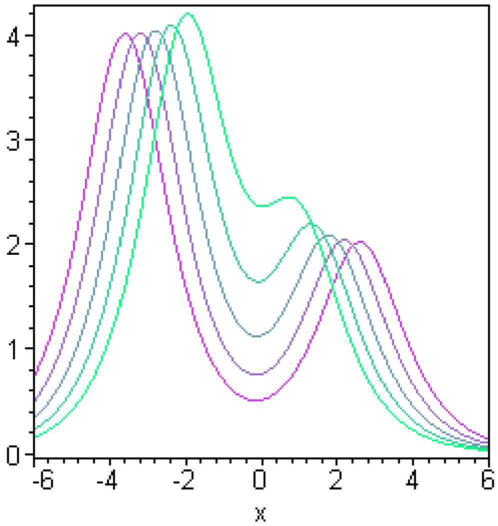


Figure 1.

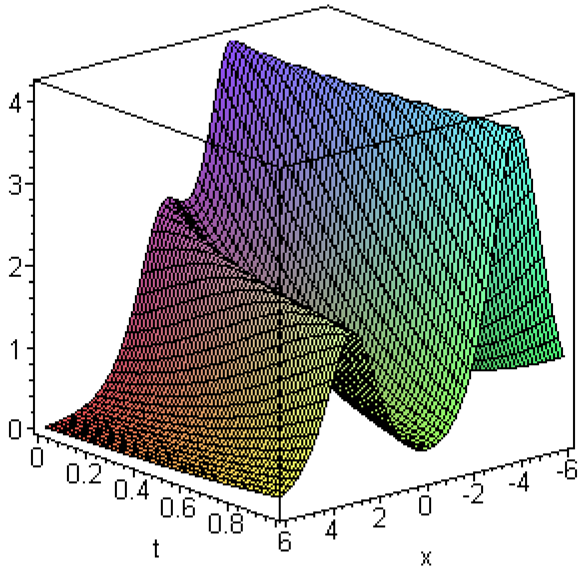
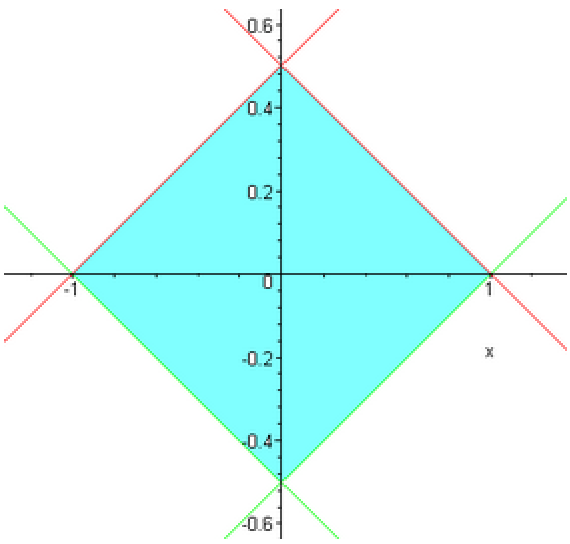


Figure 2.

In general, the superposition form of the found above general solution shows that the domain of existence (and regularity) of the solution is quite specific. Namely, if we assume that both $F(s)$ and $G(s)$ are of class C^2 in the interval $(a,b) \subset \mathbb{R}$ then $u(x,t)$ is of class C^2 in a rectangle domain

$$a < x \pm ct < b,$$

which is called *rectangles of characteristics* (see the picture below).



The initial value problem

Let us consider the following Cauchy problem for the above wave equation:

$$u(x,0)=g(x), \qquad u_t'(x,0)=h(x),$$

where g and h are the initial amplitude and velocity respectively. Using the representation (*) yields the system

$$u(x,0)=g(x)=F(x)+G(x)$$

$$u_t(x,0)=h(x)=cF'(x)-cG'(x).$$

Integrating the later equation, obtain

$$F(x)-G(x)=\frac{1}{c}\int_0^x h(s)\,ds+C$$

and combining this with the former equation we find from the obtained linear system that

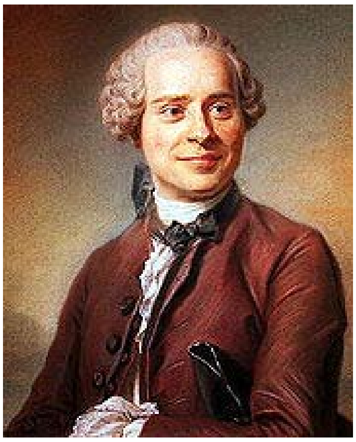
$$F(x)=\frac{1}{2}g(x)+\frac{1}{2c}\int_0^x h(s)\,ds+\frac{C}{2}$$

$$G(x)=\frac{1}{2}g(x)-\frac{1}{2c}\int_0^x h(s)\,ds-\frac{C}{2}$$

for some new constant C_1.

Substituting the found relations, this yields the celebrated **d' Alembert's formula** for the Cauchy problem of the one-dimensional homogeneous wave equation**:**

$$u(x,t)=\frac{g(x+ct)+g(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds.$$



Jean le Rond d'Alembert (1717 – 1783), a French mathematician and physicist

Some conclusions from d'Alembert's formula

- A straightforward calculation shows that d'Alembert's formula gives a C^2-solution to the above Cauchy problem provided that $g \in C^2$ and $h \in C^1$.

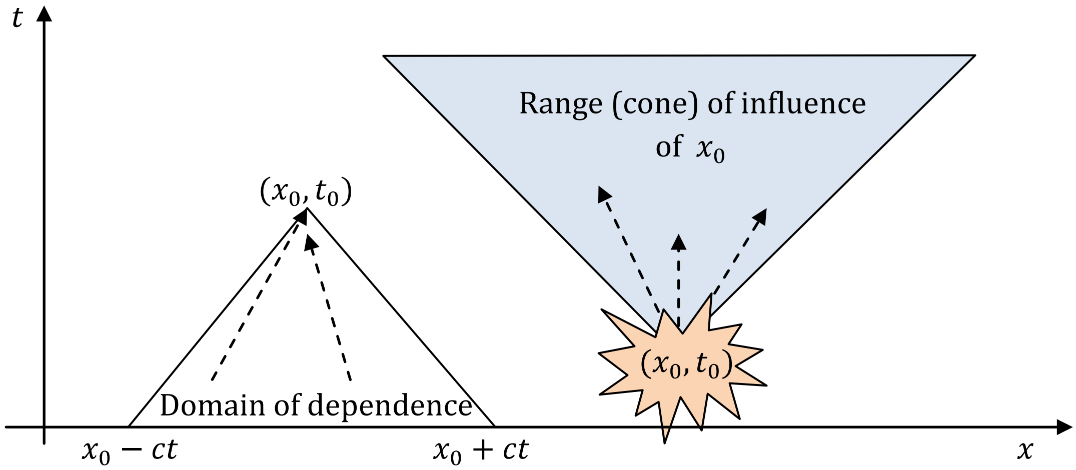
- Moreover, the smoothness of $u(x,t)$ is prescribed by that of its initial conditions, for instance, if $g \in C^{p+1}$ and $h \in C^p$ then $u(x,t)$ is a C^{p+1}-solution.

- The solution is **unique** and $u(x,t)$ depends **continuously** on the data. Hence the Cauchy problem for the wave equation is *well posed*. In other words, a small change in either g or h results in a correspondingly small change in the solution $u(x,t)$.

- One can see from the d'Alembert formula (see also the picture above) that the solution at some point (x_0, t_0), where $x_0 \in \mathbb{R}$, $t_0 \geq 0$, is completely determined by the initial data in the following interval (the *domain of dependence* for (x_0, t_0)):

$$x_0 - ct_0 \leq x \leq x_0 + ct_0$$

  Physically, this property is equivalent to the finite propagation speed of signals:



- The later property provides also the background of *special relativity* (considered firstly by Hendrik Lorentz and Henri Poincaré, and later by Albert Einstein). The cone above then is the so-called light cone in the space-time.

Parallelogram rule

Return to the general solution given above,

$$u(x,t) = F(x+ct) + G(x-ct).$$

Recall our notation $\lambda = x + ct$, $\mu = x - ct$, and consider some functions $F(\lambda)$ and $G(\mu)$ and a rectangle $ABCD$ in the $\lambda\mu$-plane as shown in Figure 3 below:

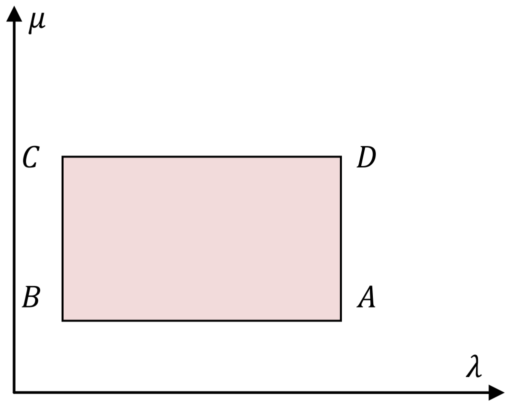


Figure 3

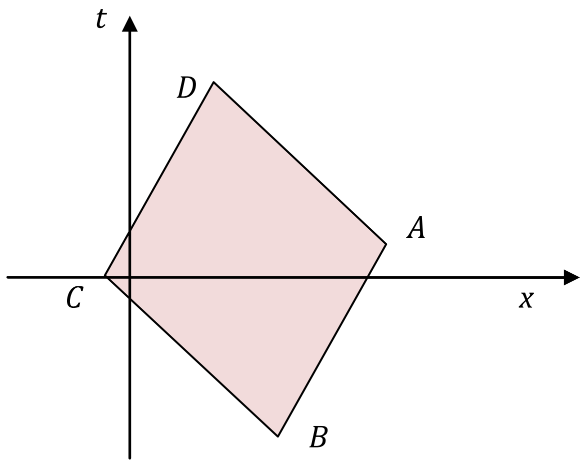


Figure 4

Since $F(\lambda)$ is constant along vertical lines and $G(\mu)$ is constant along horizontal lines, we have

$$F(A) = F(D), \quad F(B) = F(C), \quad G(A) = G(B), \quad G(C) = G(D).$$

Using our representation $u(\lambda,\mu) = F(\lambda) + G(\mu)$ we find

$$u(A) + u(C) = u(B) + u(D) \qquad (**)$$

that is the sums of the values of u at opposite vertices are equal. Translated to the xt-plane (see Figure 4), we view the previous relation as a parallelogram rule for solutions (recall that the sides of the latter parallelogram are segments of characteristics).

Application I: weak (generalized) solutions

It is natural to expect that the above formula defines a (generalized) solution when the functions F and G are no longer of class C^2. Let us assume that the initial condition $g(x)$ contains a discontinuity at some x_0 and $h(x)$ is a smooth function, say $h \equiv 0$. It follows from d'Alembert formula that

$$u(x,t) = \frac{1}{2}\big(g(x+ct) + g(x-ct)\big)$$

will be discontinuous at each point (x,t) such that $x \pm ct = x_0$, that is, at each point of the two characteristic lines intersecting at the point $(x_0, 0)$. This means that

discontinuities are propagated along the characteristic lines

At each point of the characteristic lines, the partial derivatives of the function $u(x,t)$ fail to exist, and hence, u can no longer be a solution of the Cauchy problem in the usual sense. However, such a function may be recognized as a *weak*, or a *generalized* solution of the Cauchy problem.

Definition. A *generalized solution* of the wave equation is any function $u(x,t)$ satisfying (**) for every such parallelogram in its domain.

Example 1. Let $F(x) = |x|$ and $G \equiv 0$. For simplicity we assume that $c = 1$. Then

$$u = |x-t| + |x+t|$$

is the weak solution in the sense of the definition given above. (Notice that u is just a continuous function!).

Application II: the reflection method (a semi-infinite string)

Another useful application of the above 'parallelogram rule' is the so-called ***reflection method***. We explain it for the case when the initial/boundary problem for semi-infinite vibrating string:

$$u''_{tt} - c^2 u''_{xx} = 0$$

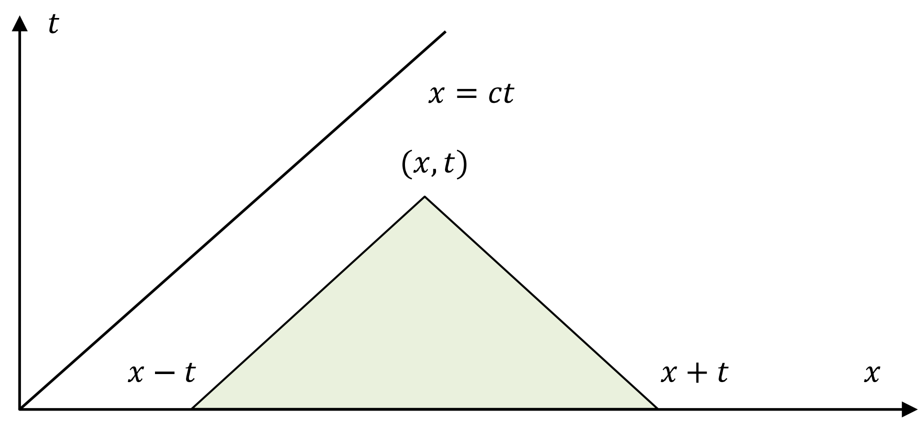
is given in the wedge $x \geq 0,\ t \geq 0$:

$$u(x,0) = g(x), \quad u'_t(x,0) = h(x), \qquad x > 0,\ t = 0,$$

$$u(0,t) = 0, \qquad x = 0, \quad t > 0$$

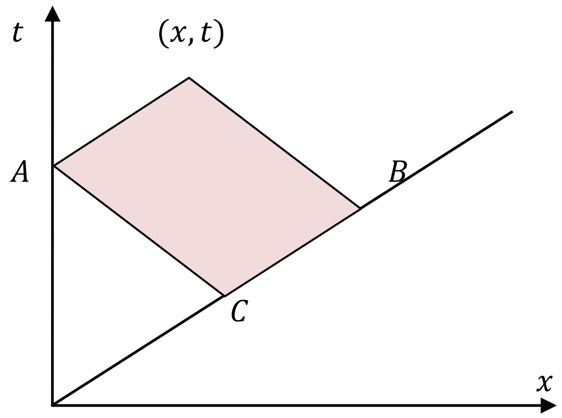
and the compatibility condition holds: $g(0) = h(0) = 0$.

Solution by the reflection method. For any point (x_0, t_0) in the **lower** infinite triangle $x \geq t \geq 0$ (see the picture below) $u(x,t)$ can be found by the d'Alembert formula by using the initial problem on interval $(x-t, x+t)$:



$$u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\, ds.$$

In the upper infinite triangle, $0 \le x \le t$ we also consider an arbitrary point (x,t) and draw characteristics from this point, and reflect one characteristic which meets the t-axis as shown in the picture:



Then by parallelogram rule we have

$$u(x,t) = u(A) + u(B) - u(C)$$

Here

$$A = (0, ct - x), \qquad B = \left(\frac{x + ct}{2}, \frac{x + ct}{2c}\right), \qquad C = \left(\frac{ct - x}{2}, \frac{ct - x}{2c}\right).$$

Hence setting $\xi_1 = \frac{x+ct}{2}$ and $\xi_2 = \frac{ct-x}{2}$, we get

$$u(x,t) = u(0, ct - x) + u\left(\xi_1, \frac{\xi_1}{c}\right) - u\left(\xi_2, \frac{\xi_2}{c}\right)$$

An taking into account the boundary condition $u(0,\tau) = 0$ $(\tau > 0)$ and that along the diagonal:

$$u\left(\xi, \frac{\xi}{c}\right) = \frac{1}{2}[g(2\xi) + g(0)] + \frac{1}{2c}\int_0^{2\xi} h(s)\, ds.$$

we obtain finally

$$u(x,t) = \frac{1}{2}[g(x + ct) - g(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\, ds\,.$$

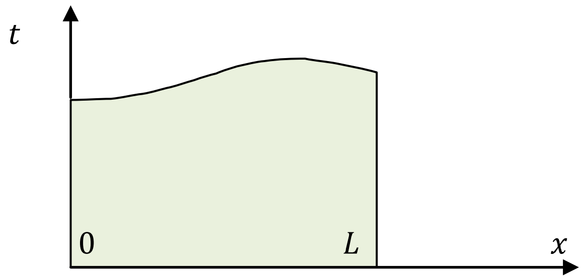
Eigenfunctions approach (a finite string)

Now we demonstrate apply the Fourier method to the Cauchy problem for a finite string, say of length L,

$$u(x,0) = g(x), \qquad u'_t(x,0) = h(x), \qquad x \in (0, L)$$

and *fixed* at both ends:

$$u(0,t) = u(L,t) = 0, \qquad t \ge 0.$$



Applying the Fourier method of separation of variables we look for a solution given by the trigonometric series (we skip here the question about convergence) below:

$$u(x,t)=\sum_{n=0}^{\infty} a_n(t)\sin\frac{n\pi x}{L}.$$

(Notice that the boundary condition is satisfied automatically). Substitution of u yields

$$a_n''(t)=-\left(\frac{n\pi c}{L}\right)^2 a_n(t),$$

hence $a_n(t)=\alpha_n\cos\frac{n\pi c}{L}t+\beta_n\sin\frac{n\pi c}{L}t$. Notice that

$$a_n(0)=\alpha_n,\qquad a_n'(0)=\frac{n\pi c}{L}\beta_n,$$

In order to determine the constants we apply the Cauchy and boundary conditions:

$$g(x)=u(x,0)=\sum_{n=0}^{\infty} a_n(0)\sin\frac{n\pi x}{L}=\sum_{n=0}^{\infty}\alpha_n\sin\frac{n\pi x}{L},$$

$$h(x)=u_t'(x,0)=\sum_{n=0}^{\infty} a_n'(t)\sin\frac{n\pi x}{L}=\frac{n\pi c}{L}\sum_{n=0}^{\infty}\beta_n\sin\frac{n\pi x}{L}.$$

Hence, expending $g(x)$ and $h(x)$ in the trigonometric series in $[0,L]$ (notice that it is reasonably to require $g(0)=g(L)=h(0)=h(L)=0$) we find α_n,β_n.

Example 2. Determine the solution fo the following problem

$$u_{tt}=u_{xx},\qquad 0<x<\pi,\ \ t>0,$$

$$u(x,0)=2\sin x-\sin 2x\,,\qquad u_t(x,0)=0,\qquad x\in[0,\pi],$$

$$u(0,t)=u(\pi,0)=0,\qquad t>0.$$

Solution. We have $g(x)=2\sin x-\sin 2x$, hence $\alpha_1=2,\alpha_2=-1$ and $\alpha_n=0, n\geq 3$. Similarly $\beta_n=0$ for all n. Hence

$$a_n(t)=\alpha_n\cos\frac{n\pi c}{L}t=\alpha_n\cos nt$$

and we obtain: $u(x,t)=a_1(t)\sin x+a_2(t)\sin 2x=2\cos t\sin x-\cos 2t\sin 2x$. ∎