Lecture 7: The wave equation, II

The nonhomogeneous wave equation

Now we consider the nonhomogeneous (NH) wave equation on the real line

$$u''_{tt} - c^2 u''_{xx} = f(x,t)$$

subject to the following initial conditions (IC): $u(x,0) = g(x)$, $u'_t(x,0) = (x)$.

Remark: Solution of the NH equation can be represented as a sum of two other solutions:

- *Problem I:* the *nonhomogeneous* wave equation $v_{tt} - c^2 v_{xx} = f$ with *homogeneous* IC:

$$v(x,0) = 0, \qquad v'_t(x,0) = 0,$$

- *Problem II:* the *homogeneous* wave equation $u_{tt} - c^2 u_{xx} = 0$ with *nonhomogeneous* IC:

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

Thus, it suffices only to consider the first problem. We apply the method due to Duhamel (*Jean Marie Constant Duhamel* (1797–1872), a French mathematician).

Namely, consider an auxiliary problem

$$U''_{tt} - c^2 U''_{xx} = 0, \qquad \text{or} \quad x \in \mathbb{R}, \quad t > 0$$

$$U(x,0,s) = 0, \qquad U'_t(x,0,s) = f(x,s), \qquad \text{for} \quad x \in \mathbb{R}, \; s > 0.$$

Here $f(x,s)$ is the right hand side in our equation given above.

Duhamel's principle. *Assume that $U(x,t,s)$ is a C^2-function of $x \in \mathbb{R}$ and $t > 0$, continuous in s, $s > 0$. If U solves the above auxiliary problem, then solution of the Problem I is given by*

$$u(x,t) = \int_0^t U(x,t-s,s)ds.$$

Proof. Apply differentiation with respect to parameter,

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} F(x,t)dt = F(b(t),t)b'(t) - F(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}F(x,t)dt.$$

This formula holds if F and F'_t are continuous. In our case $F(x,t) = U(x,t-s,s)$, thus

$$u_t(x,t) = U(x,t-t,t)\cdot 1 - U(x,t-0,0)\cdot 0 + \int_0^t \frac{\partial}{\partial t}U(x,t-s,s)ds,$$

and applying $U(x,0,s) = 0$, we find

$$u_t(x,t) = \int_0^t U_t(x,t-s,s)ds.$$

Differentiate again and apply the second initial condition:

$$u_{tt}(x,t) = U_t(x,0,t) + \int_0^t U_{tt}(x,t-s,s)ds = f(x,t) + \int_0^t U_{tt}(x,t-s,s)ds.$$

Differentiation with respect to x yields

$$u_{xx}(x,t) = \int_0^t U_{xx}(x,t-s,s)ds,$$

hence, combining the found formulas we get

$$u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t (U_{tt}(x,t-s,s) - c^2 U_{xx}(x,t-s,s))ds = f(x,t).$$ ■

Corollary. *The solution of Problem I is given by the following explicit formula*

$$u(x,t) = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\, dr.$$

Proof. Apply d' Alembert's formula.

Example 1. Find the solution of

$$u''_{tt} - u''_{xx} = x + t$$

with the initial conditions: $u(x,0) = x, u_t(x,0) = 1$.

Solution. We have $u = u_1 + u_2$, where

- u_1 is solution to $u''_{tt} - u''_{xx} = x + t, u(x,0) = 0, u_t(x,0) = 0$
- u_2 is solution to $u''_{tt} - u''_{xx} = 0, u(x,0) = x, u_t(x,0) = 1$.

We find u_1 by Duhamel principle:

$$u_1(x,t) = \frac{1}{2}\int_0^t ds \int_{x-(t-s)}^{x+(t-s)} (r+s)\, dr = \frac{1}{2}\int_0^t (t-s)(s+x)ds = \frac{t^2(t+3x)}{6}.$$

Similarly, applying d' Alembert's formula to $g = x$ and $h = 1$, we find u_2:

$$u(x,t) = \frac{x+t+x-t}{2} + \frac{1}{2}\int_{x-t}^{x+t} 1\, ds = x + t.$$

Thus, the solution is

$$u = u_1 + u_2 = x + t + \frac{t^2(t+3x)}{6}.$$

The higher-dimensional wave equation

Consider now the Cauchy problem

$$\Delta_x u(x,t) \equiv u_{x_1x_1} + \cdots + u_{x_nx_n} = c^2 u_{tt},$$

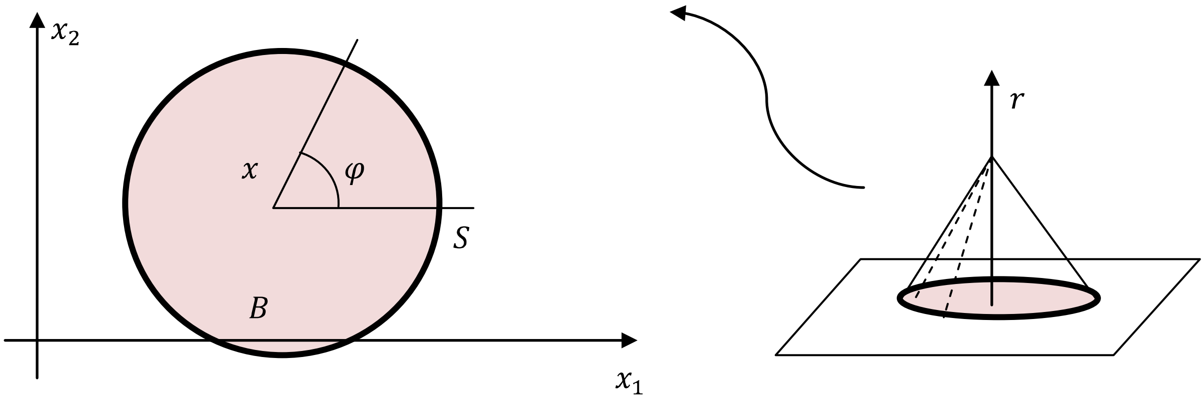
$$u(x,0) = g(x), \qquad u_t(x,0) = h(x)$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$. We shall assume, if not explicitly stated otherwise, that u is twice continuously differentiable for $x \in \mathbb{R}^n$, $t \geq 0$.

Motivation: The two-dimensional case

We demonstrate Poisson's method of spherical means by the two-dimensional case, $n = 2$. Let us denote by $x = (x_1, x_2)$ an arbitrary point and define for $r \geq 0$ and $t \geq 0$ the following average integral:

$$M_f(x,r) = \frac{1}{2\pi}\int_0^{2\pi} f(x_1 + r\cos\phi\,, x_2 + r\sin\phi)\,d\phi$$



$$M_f(x,0) = f(x).$$

Let $u(x,t) \equiv u(x_1,x_2,t)$ be the solution of the Cauchy problem and $U(x,r,t) = M_u(x,r)$. Then

$$U(x,r,0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1 + r\cos\phi\,, x_2 + r\sin\phi\,, 0)\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} g(x_1,x_2)\,d\phi = M_g(x,r)$$

$$U_t(x,r,0) = \quad \dots \quad = M_h(x,r).$$

On the other hand, substitution of $r = 0$ yields

$$U(x,0,t) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1,x_2,t)\,d\phi = u(x_1,x_2,t).$$

Hence <u>the solution can be recovered from U.</u> Differentiating U with respect to r we find

$$U_r(x,r,t) = \frac{1}{2\pi}\int_0^{2\pi}\left(u_{x_1}\cos\phi + u_2\sin\phi\right)d\phi = \frac{1}{2\pi}\int_0^{2\pi}\langle\nabla u, \nu\rangle\,\frac{ds}{r}$$

where $\nu = (\cos\phi\,, \sin\phi)$ is the outward unit normal to the circle S and $ds = r\,d\phi$ is the length element along the circle of radius r.

Applying the divergence theorem we find

$$\frac{1}{2\pi}\int_0^{2\pi} \langle \nabla u, \nu \rangle \frac{ds}{r} = \frac{1}{2\pi r}\int_B \text{ div } \nabla u \, dx_1 dx_2 = \frac{1}{2\pi r}\int_B \Delta u \, dx_1 dx_2$$

and using $\Delta u = \frac{1}{c^2} u_{tt}$ we obtain

$$U_r(x,r,t) = \frac{1}{2\pi r c^2}\int_B u_{tt} \, dx_1 dx_2 = \frac{1}{2\pi r c^2}\frac{\partial^2}{\partial t^2}\int_B u \, dx_1 dx_2$$

(B is the disk with boundary S). Write the inner integral in the polar coordinates (center at x)

$$\int_B u \, dx_1 dx_2 = \int_0^r \rho \, d\rho \int_0^{2\pi} u(x_1 + \rho\cos\phi, x_2 + \rho\sin\phi, t) \, d\phi = 2\pi \int_0^r U(x,\rho,t)\rho \, d\rho.$$

We obtain

$$rc^2 U_r(x,r,t) = \frac{\partial^2}{\partial t^2}\int_0^r U(x,\rho,t)\rho \, d\rho = \int_0^r U_{tt}(x,\rho,t)\rho \, d\rho,$$

which yields finally

$$c^2 (rU_r)_r = (rU)_{tt} \qquad \Rightarrow \qquad \frac{1}{c^2} U_{tt} = U_{rr} + \frac{1}{r} U_r.$$

The general case: Euler-Poisson equation

Define the *spherical mean* of a continuous function $f(x)$ in $\mathbb{R}^n$, $n \geq 2$, by

$$M_f(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} f(x + r\xi) \, dS_\xi$$

where ω_n denotes is the normalization constant (the area of the unit sphere in $\mathbb{R}^n$) and dS_ξ denotes the surface measure. As above, we have

$$M_f(x,0) = f(x).$$

Euler-Poisson equation. *Let u be the solution of the Cauchy problem*

$$\Delta_x u(x,t) = c^2 u_{tt}, \qquad u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

Then $U(x,r,t) \equiv M_u(x,r)$ *is twice continuously differentiable function and*

$$\frac{1}{c^2} U_{tt} = U_{rr} + \frac{n-1}{r} U_r, \qquad for \quad t > 0, \qquad r > 0$$

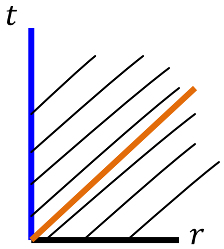
$$U(x,0,r) = M_g(x,r), \qquad U_t(x,0,r) = M_h(x,r).$$

(See **Appendix** for the proof).

Application I: The three-dimensional case

If $n = 3$ the Euler-Poisson equation takes the form

$$\tfrac{1}{c^2} U_{tt} = U_{rr} + \tfrac{2}{r} U_r,$$



Setting $v(x,r,t) = rU(x,r,t)$ we arrive at the one-dimensional wave equation:

$$v_{tt} - c^2 v_{rr} = 0.$$

Translating the initial conditions we find

$$v(x,r,0) = rM_g(x,r), \qquad v_t(x,r,0) = rM_h(x,r), \qquad v(x,0,t) = 0.$$

Recall also that in order to recover u we set $r = 0$:

$$u(x,t) = U(x,0,t) = \lim_{r\to 0} \frac{v(x,r,t)}{r} = (\text{l'Hopital's rule}) = v_r(x,0,t).$$

Hence we need to determine v for ***small*** r. To this end we apply the d' Alembert formula for *semi-infinite* string (*obs.*!: with respect to r and t) for **the upper-triangle**

$$v(x,r,t) = \frac{1}{2}\Big((r+ct)M_g(x,r+ct) - (ct-r)M_g(x,r-ct)\Big) + \frac{1}{2c}\int_{ct-r}^{ct+r} \rho M_h(x,\rho)d\rho.$$

This yields the **Kirchhoff formula**

$$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1}^{t} g(x+ct\xi)dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1}^{t} h(x+ct\xi)dS_\xi$$

From this formula we infer:

- the Cauchy problem in $n = 3$ is well-posed
- the domain of dependence of a point $A(x,y,z,t)$ with $t > 0$ is the *sphere* with center at $(x,y,z) \in \mathbb{R}^3$ and of radius ct. In contrast with the 1-dimensional case, we have a different phenomenon: not a **ball**, but its **boundary** (sphere) has influence on the point A.

Example 2. We demonstrate the above method by solving the following Cauchy problem:

$$u_{xx} + u_{yy} + u_{zz} = u_{tt},$$

$$u(x,y,0) = xy, \qquad u_t(x,y,0) = 2x^2 - z^2.$$

Solution. The first average is

$$M_g = \frac{1}{4\pi}\int_S (x + r\xi_1)(y + r\xi_2)\, dS_\xi = xy,$$

because the average of any combination $\xi_1^k \xi_2^m \xi_3^l$ with at least one odd exponent must be zero.

The same argument yields

$$M_h = \frac{1}{4\pi}\int_S [2(x + r\xi_1)^2 - (z + r\xi_3)^2]\,dS_\xi = 2x^2 - z^2 + \frac{1}{4\pi}\int_S (2\xi_1^2 - \xi_3^2)\,dS_\xi.$$

The latter integral can be found by the following useful trick. Denote by a_i the average of ξ_i^2 Then by symmetry, $a_1 = a_2 = a_3$ and

$$3a_1^2 = a_1^2 + a_2^2 + a_3^2 = \frac{1}{4\pi}\int_S (\xi_1^2 + \xi_2^2 + \xi_3^2)\,dS_\xi = \frac{1}{4\pi}\cdot r^2 \cdot 4\pi = r^2,$$

hence

$$\frac{1}{4\pi}\int_S (2\xi_1^2 - \xi_3^2)\,dS_\xi = 2a_1^2 - a_2^2 = \frac{r^2}{3}.$$

We have $M_h = 2x^2 - z^2 + \frac{r^2}{3}$. Hence we find the auxiliary function v:

$$v(x, r, t) = \frac{1}{2}\big((r + t)xy - (t - r)xy\big) + \frac{1}{2}\int_{t-r}^{t+r} \rho\left(\frac{\rho^2}{3} + 2x^2 - z^2\right)d\rho$$

$$= rxy + \frac{rt}{3}(r^2 + t^2 + 6x^2 - 3z^2).$$

Differentiating the last equality w.r.t. r we find u:

$$u(x, y, t) = v_r|_{r=0} = xy + \frac{t^3}{3} + t(2x^2 - z^2).$$

■

Conservation of energy

Energy of a function is the following integral (if well defined)

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2 + c^2|\nabla u|^2)\,dx.$$

Otherwise, we set $E \equiv \infty$. Differentiating the integral yields

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^n} u_t u_{tt} + c^2\langle \nabla u, \nabla u_t\rangle \ = \text{integrating by parts}$$

$$= \int_{\mathbb{R}^n} (u_t u_{tt} - c^2 u_t \operatorname{div} \nabla u)\,dx = 0$$

It follows that $E(t) = const$.

Remark. This implies **uniqueness** because if u_1 and u_2 are two solutions to one Cauchy problem then $u_1 - u_2$ has zero initial energy.

Appendix *: Derivation of Euler-Poisson- equation

Proof. The fact that $U(x,r,t)$ is two times continuously differentiable in t follows from standard facts on differentiability of an integral depending on parameters. The situation with parameter r requires however a further analysis because the set $B_x(r)$ itself depends on r.

Step 1: We show that $U(x,r,t)$ is continuously differentiable in r and

$$U_r(x,r,t) = \frac{1}{\omega_n r^{n-1}} \int_{B_x(r)} \Delta_y u(y,t)\ dy.$$

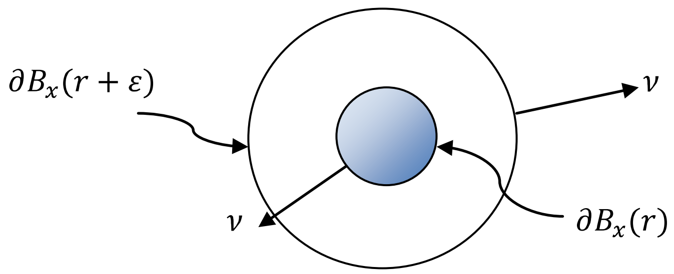
We consider an auxiliary function

$$f(r) = \omega_n r^{n-1} U(x,r,t) = \int_{\partial B_x(r)} u(y,t)\, dS_y$$

For $\varepsilon > 0$ we have

$$f(r+\varepsilon) - f(r) = \int_{\partial B_x(r+\varepsilon)} u(y,t)\, dS_y - \int_{\partial B_x(r)} u(y,t)\, dS_y = \int_{\partial V} u(y,t)\, dS_y$$

where $V := B_x(r+\varepsilon)\backslash\overline{B_x(r)}$ is the spherical shell with the boundary equipped with outward normal, $\partial V = \partial B_x(r+\varepsilon) - \partial B_x(r)$, see the picture below:



The unit vector-field $F(y) := \frac{y-x}{|y-x|}$ coincides with the gradient of the distance function (check!)

$$F(y) = \frac{y-x}{|y-x|} = \nabla_y |y-x|.$$

Moreover, the divergence of this vector-field is

$$\mathrm{div}_y F(y) = \mathrm{div}_y \frac{y-x}{|y-x|} = \sum_{k=1}^{n} \frac{\partial}{\partial y_k}\left(\frac{y_k - x_k}{|y-x|}\right) = \frac{n}{|y-x|} - \sum_{k=1}^{n} \frac{(y_k - x_k)^2}{|y-x|^3} = \frac{n-1}{|y-x|}.$$

On the other hand, the unit normal vector at $y \in \partial B_x(s)$ is

$$\nu = \frac{y-x}{s} \equiv F(y).$$

In particular: $\nu \cdot F(y) = \nu \cdot \nu = F(y) \cdot F(y) = 1$. Applying the divergence theorem we obtain

$$f(r+\varepsilon) - f(r) = \int_{\partial V} 1 \cdot u(y,t)\, dS_y = \int_{\partial V} \nu \cdot F(y) u(y,t)\, dS_y = \int_V \mathrm{div}_y \left(F(y) u(y,t)\right) dy.$$

Applying Fubini's theorem we find

$$f(r+\varepsilon)-f(r)=\int_{r}^{r+\varepsilon}d\xi\int_{\partial B_x(\xi)}\operatorname{div}_y\,(F(y)u(y,t))\;dS_y$$

and it follows easily that the following limit exists

$$\lim_{\varepsilon\to 0+}\frac{f(r+\varepsilon)-f(r)}{\varepsilon}=\int_{\partial B_x(r)}\operatorname{div}_y\,(F(y)u(y,t))\;dS_y$$

(and similarly for the left limit). Thus $f(r)$ is (continuously) differentiable and

$$f^{'}(r)=\int_{\partial B_x(r)}\operatorname{div}_y\,(F(y)u(y,t))\;dS_y.$$

Next, apply the above formula for div $F(y)$ and recall that the normal ν coincides with $F(y)$:

$$\operatorname{div}_y\,\big(F(y)u(x,t)\big)=u(x,t)\operatorname{div}_y F(y)+F(y)\cdot\nabla_y u(y,t)=u(y,t)\frac{n-1}{|y-x|}+\nu\cdot\nabla_y u(y,t)$$

Applying the divergence theorem again, we get

$$f^{'}(r)=\int_{\partial B_x(r)}\left(u(y,t)\frac{n-1}{|y-x|}+\nu\cdot\nabla_y u(y,t)\right)dS_y=\frac{n-1}{r}f(r)+\int_{B_x(r)}\Delta_y u(y,t)\;dy.$$

On the other hand,

$$f^{'}(r)-\frac{n-1}{r}f(r)=r^{n-1}(r^{1-n}f(r))'$$

Recalling the definition of $U(x,r,t)$ we obtain

$$\int_{B_x(r)}\Delta_y u(y,t)\;dy=r^{n-1}(r^{1-n}f(r))'\equiv\;\omega_n r^{n-1}U_r^{'}(x,r,t)\,.$$

It follows also from the above argument that differentiability of $f^{'}(r)$ is equivalent to that of integral $\int_{B_x(r)}\Delta_y u(y,t)\;dy$. But this integral is differentiable because the Laplacian $\Delta_y u(y,t)$ is a continuous function in all parameters.

Step 2. We return to our formula for $U_r^{'}(x,r,t)$ and apply to it our wave equation:

$$\omega_n r^{n-1}U_r^{'}(x,r,t)=\int_{B_x(r)}\Delta_y u(y,t)\;dy=\int_{B_x(r)}u_{tt}^{''}(y,t)\;dy$$

Differentiation of the latter identity w.r.t. r yields

$$\left(\omega_n r^{n-1}U_r^{'}(x,r,t)\right)_r^{'}=\frac{\partial}{\partial r}\int_{B_x(r)}u_{tt}^{''}(y,t)\;dy=\int_{\partial B_x(r)}u_{tt}^{''}(y,t)\;dy=\frac{\partial^2}{\partial t^2}\int_{\partial B_x(r)}u(y,t)\;dy$$

which implies the desired equation: $(r^{n-1}U_r^{'})_r^{'}=(r^{n-1}U)_{tt}^{''}=r^{n-1}U_{tt}^{''}$. The initial conditions $U(x,r,0)=M_g(x,r)$ and $U_t^{'}(x,r,0)=M_{h_h}(x,r)$ follow now from the definition. QED.