Lecture 8: The Laplace and Poisson equations

Now we consider *boundary-value problems*.

- Mathematically, a boundary-value problem is finding a function which satisfies a given partial differential equation and particular boundary conditions.
- Physically speaking, the problem is *independent of time*, involving only space coordinates.

Just as initial-value problems are associated with hyperbolic PDE, boundary value problems are associated with PDE of elliptic type. In contrast to initial-value problems, boundary-value problems are considerably more difficult to solve.

The main model example of an elliptic type PDE is the **Laplace equation**

$$\Delta u(x) \equiv u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0,$$

where $x = (x_1, x_2, \dots, x_n) \in \Omega$ and Ω is a domain in $\mathbb{R}^n$. Solutions of this equation are called *harmonic functions.* This equation can also be thought as the wave equation

$$\Delta u(x) = \frac{1}{c^2} u_{tt}$$

with infinite `sound velocity' c.

There are two main modifications of the Laplace equation: the **Poisson equation** (a non-homogeneous Laplace equation):

$$\Delta u(x) = f(x), \qquad x \in \Omega,$$

and the **eigenvalue problem** (the **Helmholtz equation**)

$$\Delta u(x) = \lambda u, \qquad \lambda \in \mathbb{R}.$$

All above equations are of elliptic type (there are no characteristics).

Definition. A function is said to be ***harmonic*** in a domain Ω if it satisfies the Laplace equation and if it and its first two derivatives are continuous in Ω.

Variational derivation of the Laplace equation

Derivation of an equation which contains the Laplacian (Laplace operator Δ) usually is concerned with minimization problems. For instance, if we minimize the Dirichlet integral

$$I_f(u) := \int_\Omega (|\nabla u|^2 + f u^2)\, dx,$$

among all smooth enough functions $v(x)$ with given boundary values, say

$$v(x) = g(x), \qquad g\colon \partial\Omega \to \mathbb{R}, \tag{1}$$

then we arrive at the Dirichlet problem for the Poisson equation. We give below an heuristic argument how to derive the Laplace equation (that is if $f \equiv 0$).

A non-trivial component of any variational problem is to establish the existence of solution. If we know, however, that a solution does exist we can apply the standard variational argument which dates back to Pierre de Fermat. In our case, let us assume that u is a solution, that is $I_0(u) \leq I_0(v)$ for `any' v such that the boundary condition (1) is satisfied. This means that for any $t \in \mathbb{R}$ and any function $w(x)$ with zero boundary data the linear combination

$$v(x) = u(x) + tw(x)$$

satisfies (1). Then the above inequality yields for any $t \in \mathbb{R}$

$$I_0(u) \equiv \int_\Omega |\nabla u|^2 \, dx \leq I(v) \equiv \int_\Omega \left(|\nabla u|^2 + 2t\langle \nabla w, \nabla u\rangle + t^2 |\nabla w|^2\right) dx$$

Simplifying we find

$$t \int_\Omega \left(2\langle \nabla w, \nabla u\rangle + t|\nabla w|^2\right) dx \geq 0, \qquad \forall \, t.$$

This inequality immediately yields: $\int_\Omega 2\langle \nabla w, \nabla u\rangle \, dx = 0$, and applying the divergence theorem together with the zero boundary condition ($w = 0$ on $\partial\Omega$) we obtain

$$\int_\Omega w \Delta u \, dx = 0, \qquad \forall \, w.$$

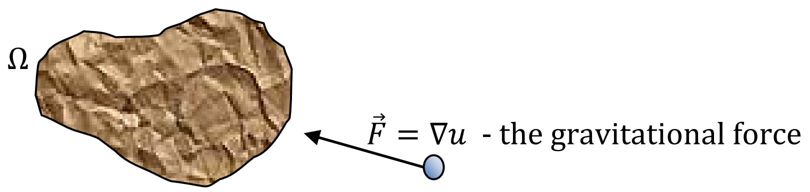
The equality $\Delta u \equiv 0$ in Ω follows now from the fundamental lemma of calculus of variations.

Remark. When $n = 2$, any harmonic function (locally) is the real part of some holomorphic function. This fact has many applications in hydrodynamics.

Potential theoretic interpretations

Similar, if we minimize $I_f(v, \Omega)$ among all $v(x)$ which satisfy (1), we obtain the Poisson equation. We briefly mention the potential interpretation of the Poisson equation.

The function $-f(x)/\omega_n$ (ω_n denotes as usually the area of the unit sphere in $\mathbb{R}^n$) is the density of masses distributed in the body Ω produced the potential $u(x)$ inside the body:



Outside the body, the potential is a harmonic function – a solution of a homogeneous Laplace equation. Similarly, one can consider a potential electric field with the gradient $u(x)$ and the charge density $-f(x)/\omega_n$.

The well-known *inverse problem* in potential theory asks whether there exist two different bodies in $\mathbb{R}^3$ having the same potentials outside the union of bodies.

Separation of variables

Problem 1. Solve the two-dimensional Laplace equation

$$u''_{xx} + u''_{yy} = 0$$

in the rectangle $\Omega = \{(x, y):\ 0 \le x \le a,\ 0 \le y \le b\}$ with the boundary conditions:

$$u(0, y) = u(a, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, b) = g(x),$$

where $0 \le x \le a,\ 0 \le y \le b$.

Solution. It is natural to apply the Fourier method of separation of variables described earlier for the wave equation. Namely, consider solutions given by the ansatz

$$u(x, y) = v(x)w(y).$$

This implies $v''(y)w(y) + v(x)w''(y) = 0$ and due to independency of x and y we find that

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = C$$

where C is some constant. Then the boundary conditions above imply

$$v(0) = v(a) = w(0) = 0, \qquad v(x)w(b) = g(x).$$

Consider the first equation: since $v'' - Cv = 0$ and $v(0) = v(a) = 0$, we have either $v \equiv 0$ or $C < 0$, say $C = -c^2$. The resulting equation $v'' + c^2 v = 0$ gives then a series of solutions with zero-boundary condition:

$$v_n(x) = \sin\frac{\pi n x}{a}, \qquad n = 1,2,3, \dots$$

The corresponding $c_n = \frac{\pi n}{a}$. Using the found C_n, we have for the second component

$$w'' - c_n^2 w = 0,$$

that is the solution is found by means of the hyperbolic functions:

$$w(y) = Ae^{c_n y} + Be^{-c_n y} = A' \cosh c_n y + B' \sinh c_n y.$$

The boundary condition $w(0) = 0$ yields $A' = 0$. In summary, we have the series of solutions

$$u = A_n \sin\frac{\pi n x}{a} \sinh\frac{\pi n y}{a}, \qquad n = 1,2, \dots$$

Let us assume that our solution can be found as a (formal) sum of the ansatz-solutions found above, that is

$$u(x,y)=\sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{a}\sinh\frac{\pi n y}{a}$$

Applying the remained boundary condition $v(x)w(b)=g(x)$ we find

$$g(x)=\sum_{n=1}^{\infty}\left(A_n \sinh\frac{\pi n b}{a}\right)\sin\frac{\pi n x}{a}$$

The numbers $A_n \sinh\frac{\pi n b}{a}$ now can be found by the Fourier expansion of $g(x)$.

Problem 2. Let $\Omega=\{(x,y):\ x^2+y^2<1\}$ be the unit disk. Solve the following problem:

$$u_{xx}+u_{yy}=0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu}=h \ \text{ on } \partial\Omega\,.$$

Solution. Here $h=h(\theta)$ is some continuous function on the unit circle (where θ is the polar angle, i.e. $\tan\theta=\frac{y}{x}$). Rewriting the Laplacian in the polar coordinates yields (verify this!)

$$\Delta u=u''_{rr}+\frac{1}{r}u'_r+\frac{1}{r^2}u''_{\theta\theta}=0$$

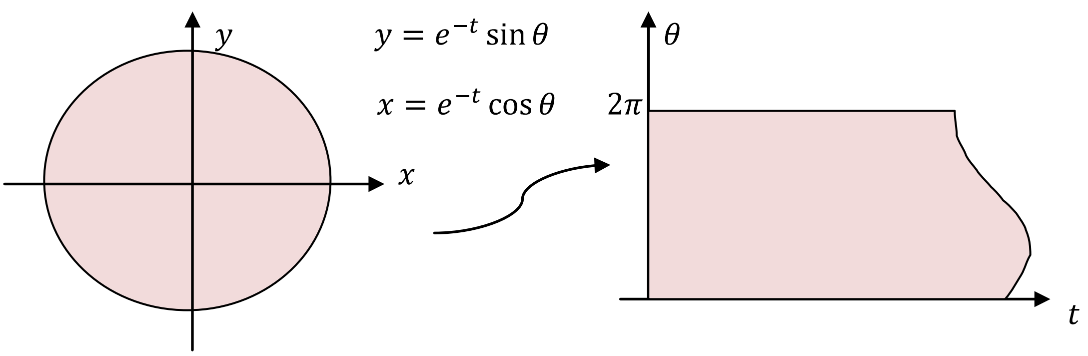
for $0\le\theta\le 2\pi$ and $0\le r<1$. Here $r=\sqrt{x^2+y^2}$ is the polar radius. The boundary condition takes then the form

$$\frac{\partial u}{\partial \nu}=u'_x\cos\theta+u'_y\sin\theta=u_r \quad\Rightarrow\quad \frac{\partial u}{\partial \nu}=u'_r|_{r=1}=h(\theta).$$

The new change of variables, $r=e^{-t}$, readily yields

$$r^2\Delta u=r^2u''_{rr}+ru'_r+u''_{\theta\theta}=u''_{tt}+u''_{\theta\theta}=0$$

This reduces our original boundary problem in the disk to the rectangular boundary problem:



Applying the product form: $u(x,y)=v(t)w(\theta)$ we see that $w(\theta)$ must be 2π-periodic, because we are looking for a *continuous* solution. Thus

$$v''(t)w(\theta)+v(t)w''(\theta)=0 \quad\Rightarrow\quad \frac{v''(x)}{v(x)}=-\frac{w''(y)}{w(y)}=C.$$

The constant C must be positive to be consistent with the fact that $w(\theta)$ is a *periodic* function. Taking into account that $w(\theta)$ is actually 2π-periodic, we write

$$w(\theta) = w_n(\theta) \equiv A_n \sin\, n\theta + B_n \cos n\,\theta\,, \qquad n = 0,1,2,\ldots$$

We find then $C = n^2$ and this yields a relation for v: $v''(x) - n^2 v(x) = 0$. We have

$$v(t) = v_n(t) = a_n\, \sinh\, nt\, + b_n \cosh nt\,, \qquad n = 0, 1, 2, \ldots$$

Since

$$\sinh\, nt\, = \frac{1}{2}(e^{nt} - e^{-nt}) = \frac{1}{2}(r^n - r^{-n}), \qquad \cosh\, nt\, = \frac{1}{2}(r^n + r^{-n})$$

we obtain the final form of solution:

$$u_n(x, y) = v_n(t) w_n(\theta) = (c_n r^n + d_n r^{-n})(A_n \sin\, n\,\theta + B_n \cos n\,\theta).$$

On the other hand, taking into account that the desired solution must be **continuous** in the unit disk, we conclude that all $d_n = 0$, hence

$$u_n(x, y) = r^n(C_n \sin\, n\,\theta + D_n \cos n\,\theta).$$

This yields

$$u(x, y) = \sum_{n=0}^{\infty} r^n(C_n \sin\, n\,\theta + D_n \cos n\,\theta). \qquad (*)$$

In order to determine C_n and D_n, we apply the boundary condition:

$$u'_r|_{r=1} = \sum_{n=1}^{\infty} nC_n \sin\, n\,\theta + nD_n \cos n\,\theta = h(\theta)$$

Hence nC_n and nD_n are the Fourier coefficients of $h(\theta)$. Notice that it follows from the above formula that $h(0) = 0$.

Remark 1. Applying Abel's theorem on power series, we conclude that the series $(*)$ converges in the open unit disk.

Some corollaries

- We encounter here a new phenomenon: as it is seen from the previous examples, the Laplace equation in a bounded domain may be *overdetermined*, that is one can not specify both $u|_{\partial\Omega}$ and $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ along the boundary.
- The first type of boundary problem, $u|_{\partial\Omega} = g$, is called the *Dirichlet* problem, the second, $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = h$, is called the *Neumann* problem
- Sometimes one considers a mixed problem, by prescribing both the Dirichlet and the Neumann data on some pieces of the boundary.

Green's identities

First we recall two simple corollaries of the divergence theorem for functions $u, v \in C^2(\overline{\Omega})$ where Ω is a bounded open set in $\mathbb{R}^n$ with piecewise smooth boundary. Namely,

$$\int_{\partial\Omega} v \, \frac{\partial u}{\partial \nu} \, dS = \int_{\Omega} (v \, \Delta u + \nabla u \cdot \nabla v) \, dx$$

and its skew-symmetric analogue:

$$\int_{\partial\Omega} v \, \frac{\partial u}{\partial \nu} - u \, \frac{\partial v}{\partial \nu} \, dS = \int_{\Omega} (v \, \Delta u - u \, \Delta v) \, dx.$$

Now we list some important corollaries of the Green identities.

Theorem (Uniqueness of solution of the Dirichlet problem). *Let u_1 and u_2 be harmonic functions with equal boundary values: $u_1 = u_2$ on $\partial\Omega$, where Ω is some bounded open set. Then $u_1 \equiv u_2$ in Ω.*

Proof. Let $\Delta u = 0$ in Ω. Then substitution of $v = u$ into the first Green's identity implies

$$\int_{\partial\Omega} u \, \frac{\partial u}{\partial \nu} \, dS = \int_{\Omega} |\nabla u|^2 \, dx.$$

Observe that the latter integral is strictly positive unless u is a constant. Set $u = u_1 - u_2$, then $u = 0$ on the boundary of Ω. Hence the left hand side of the above integral identity is zero. It follows that $|\nabla u| = 0$ in Ω, hence $u = const = C$. But $u = 0$ on $\partial\Omega$, hence $C = 0$. The theorem is proved. ■

Remark 2. If one applies the above argument to the Neumann problem, then the constant C in the proof may not be zero. Thus the Neumann problem determines the solution uniquely *up to an additive constant*.

Moreover, taking $v = 1$ in the first Green's identity we obtain the following

Theorem (Necessary condition for the Neumann condition). Let u be a harmonic function with the Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = h$. Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS = 0,$$

i.e. the average of h over the boundary equals zero: $\int_{\partial\Omega} h \; dS = 0$.

Remark 3. The latter theorem shows that the Neumann condition h can't be chosen arbitrarily (cf. Problem 2 above, $h(0) = 0$).