Lecture 9: The fundamental solution

The co-area formula

Let $D \subset \mathbb{R}^n$ be a bounded domain and $f(x)$ be a function of class $C^1(D)$. Denote by $D_f(t) = \{x \in D\colon f(x) = t\}$ the t-level set of f. Let $h(x)$ be a continuous function in D. Then

$$\int_D h(x)dx = \int_{-\infty}^{+\infty} dt \int_{D_f(t)} \frac{h(y)}{|\nabla f(y)|} dS_y,$$

where dS_y is the surface measure on $D_f(t)$. In fact, the exterior integral should be taken only over the interval of those t for which $D_f(t)$ is non-empty.

Example 1 (Spherical Fubini theorem). When $f(x) = r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the distance function we have

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \Rightarrow \quad \nabla f(x) = \nabla r = \frac{x}{r} \quad \Longrightarrow \quad |\nabla f(x)| = 1.$$

Chose the domain in the co-area formula

$$D = B_0(R) = \{x \in \mathbb{R}^n\colon |x| < R\},$$

the n-dimensional ball centered at x of radius R. In this case $D_f(t)$ is non-empty for $0 \le t < R$ and the level sets are spheres $\partial B_0(t)$. Then applying the co-area formula gives

$$\int_{B_0(r)} h(x)dx = \int_0^R dt \int_{\partial B_0(t)} h(x)\, dS_x.$$

Example 2. (The volume and the area of an n-dimensional ball).We apply the previous result to $h \equiv 1$, we find the volume of the ball of radius R:

$$|B_0(R)| \equiv \int_{B_0(r)} 1 dx = \int_0^R dt \int_{\partial B_0(t)} 1\, dS_x.$$

Observe that $|B_0(R)| = |B_0(1)| \cdot R^n$ and

$$|\partial B_0(t)| = |\partial B_0(1)| \cdot t^{n-1} = \omega_n t^{n-1},$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the area of the unit sphere in $\mathbb{R}^n$. Then

$$|B_0(1)| \cdot R^n = \int_0^R \omega_n t^{n-1} dt = \frac{\omega_n}{n} R^n.$$

Therefore we have

$$\boxed{|B_0(1)| = \frac{\omega_n}{n}, \qquad |B_0(R)| = \frac{\omega_n}{n} R^n.}$$

Radial symmetric solutions of the Laplace equation

We start with finding a radial symmetric solution u of the Laplace equation in $\mathbb{R}^n$, $u(x) = v(r)$, where $r = |x|$. We have the following idenitites:

$$\frac{\partial u}{\partial x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r},$$

$$\frac{\partial^2 u}{\partial x_i \partial x_i} = \frac{v''(r)x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), \qquad i = 1, \dots, n$$

Therefore

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

Integrating this equation yields the following property.

Radial symmetric harmonic functions:

$$v(x) = \begin{cases} a \ln |x|, & n = 2 \\ \dfrac{a}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

Theorem (The Mean-value property). *Let u be a harmonic function in the closed disk $B_a(R)$. Then*

$$\frac{1}{|\partial B_a(R)|} \int_{\partial B_a(R)} u(x)\, dS_x = \frac{1}{|B_a(R)|} \int_{B_a(R)} u(x)\, dx = u(a).$$

Proof. We consider here the case $n \geq 3$. Without loss of generality we can assume that $a = 0$. Let $\varepsilon < R$ and apply the second Green's identity with u and

$$v(x) = \frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \equiv w(|x|), \qquad w(t) = \frac{1}{t^{n-2}} - \frac{1}{R^{n-2}},$$

for the domain $\Omega = B_0(R) \backslash \overline{B_0(\varepsilon)}$. Then $v(x)$ is harmonic in Ω, radially symmetric and vanishes on the exterior boundary: $v(x) = 0$ on $\partial B_0(R)$. Therefore

$$0 = \int_{\partial V} v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = -w(\varepsilon)\int_{\partial B_0(\varepsilon)} \partial_\nu u\, dS - w_r(R)\int_{\partial B_0(R)} u\, dS + w_r(\varepsilon)\int_{\partial B_0(\varepsilon)} u\, dS.$$

Here $\partial\Omega = \partial B_0(R) - \partial B_0(\varepsilon)$ (all the boundaries are oriented by the outward normal) and we used the expression for the normal derivative on the sphere of radius $t = |x|$:

$$\partial_r w(t) = \partial_\nu v(x).$$

Since

$$\partial_r w(R) = -\frac{n-2}{R^{n-1}}, \qquad \partial_r w(\varepsilon) = -\frac{n-2}{\varepsilon^{n-1}}$$

we obtain

$$\left(\frac{1}{R^{n-2}} - \frac{1}{\varepsilon^{n-2}}\right) \int_{\partial B_0(\varepsilon)} \partial_\nu u \, dS_x = -\frac{n-2}{R^{n-1}} \int_{\partial B_0(R)} u \, dS_x + \frac{n-2}{\varepsilon^{n-1}} \int_{\partial B_0(\varepsilon)} u \, dS_x. \tag{1}$$

Notice that

$$\left| \int_{\partial B_0(\varepsilon)} \partial_\nu u \, dS \right| \leq \int_{\partial B_0(\varepsilon)} |\partial_\nu u| \, dS \leq M \omega_n \varepsilon^{n-1} = O(\varepsilon^{n-1}), \qquad \varepsilon \to 0,$$

where M is the maximum value of $|\partial_\nu u|$ in a small ball. Hence the left hand side of (1) converges to zero as $\varepsilon \to 0$. Since $|\partial B_0(R)| = \omega_n R^{n-1}$, it follows then

$$\frac{1}{\omega_n R^{n-1}} \int_{\partial B_0(R)} u \, dS_x = \lim_{\varepsilon \to 0} \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_0(\varepsilon)} u \, dS_x = u(0).$$

Since $\omega_n R^{n-1} = |\partial B_0(R)|$, we have proved the first identity

$$\frac{1}{|\partial B_0(R)|} \int_{\partial B_0(R)} u\,(x) \; dS_x = u(0).$$

Rewrite this identity for $R = t$ as

$$\omega_n t^{n-1} u(0) = \int_{\partial B_0(t)} u\,(x) \; dS_x.$$

Integrating then between 0 and R and applying the spherical Fubini's theorem yields

$$\frac{\omega_n R^n u(0)}{n} = \int_0^R dt \int_{\partial B_0(t)} u \, dS_x = \int_{B_0(R)} u \, dx.$$

It remains only to notice $|B_0(R)| = \frac{\omega_n R^n}{n}$. ■

An alternative definition of harmonicity. The mean-value property is also used as an equivalent definition of harmonicity. Namely, a function u defined in a domain Ω and integrable there is called harmonic if for any ball $B_x(t) \subset \Omega$

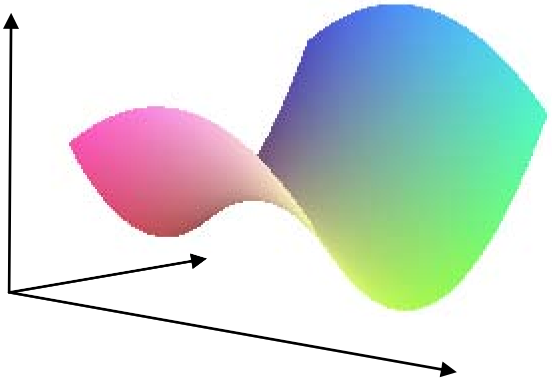
$$u(x) = \frac{1}{|B_x(t)|} \int_{B_x(t)} u\,(y) \; dy.$$

Theorem (The strong maximum principle). *Suppose $u(x)$ is harmonic inside of a connected bounded open set Ω and continuous in the closed domain $\bar{\Omega}$. Then*

$$\max_{\Omega\cup\partial\Omega} u(x) = \max_{\partial\Omega} u(x)$$

Furthermore, if there is a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\Omega} u(x)$, then $u \equiv const.$

Remark. Geometrically, the maximum principle says that the graph of a harmonic function is a saddle surface:



Proof. Notice that u is continuous in the closed domain, hence it attains its maximum value there. Denote the maximum by

$$M = \max_{\bar{\Omega}} u(x).$$

Assume now that $\max_{\partial\Omega} u(x) < M$. Then the maximum value must attained at some *interior* point, say $x_0 \in \Omega$. We have $u(x_0) = M$. Now we consider a ball $B_{x_0}(r)$ of a small enough radius such that the ball is contained in Ω. Applying the mean value property we obtain

$$M = u(x_0) = \frac{1}{|B_{x_0}(r)|} \int_{B_{x_0}(r)} u\,(y)\ dy \le M$$

But this bilateral inequality implies $u \equiv M$ in $B_{x_0}(r)$, in particular, the set E of all points satisfying $u(x_0) = M$ is *open*. On the other hand, this set E is *closed* because u is a continuous function in the ball. Since Ω is connected, we conclude that $E = \Omega$, hence $u \equiv M$ in the whole Ω. But this contradicts our assumption (recall again that u is continuous in the closed domain). This proves that the maximum is attained on the boundary: $\max_{\partial\Omega} u(x) = M$.

The second assertion follows easily from the above argument. The theorem is proved. ∎

Corollary 1 (Uniqueness) *Let $u, v \in C^2(\Omega) \cap C(\Omega)$ and $u - v = 0$ on the boundary $\partial\Omega$. Then $u \equiv v$ in Ω.*

Corollary 2 (Uniqueness of the Dirichlet problem) *Let $g(x) \in C(\partial\Omega)$ and $f(x) \in C(\Omega)$. Then there exists at most one solution $(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ of the boundary-value problem*

$$\Delta u(x) = f(x), \qquad x \in \Omega,$$

$$u(x) = g(x), \qquad x \in \partial\Omega.$$

The fundamental solution

The radial symmetric harmonic function

$$\Psi(x) \equiv \Psi_n(x) = \begin{cases} \frac{1}{2\pi} \ln |x| & n = 2 \\ -\frac{1}{(n-2)\omega_n |x|^{n-2}} & n \geq 3 \end{cases}$$

is called the ***fundamental solution*** of the Laplace equation. The reason why namely this normalization is chosen becomes clear later. Notice also that $\Psi(x)$ has a singularity at the origin.

Theorem (The characteristic property of the fundamental solution). *Let $u(x)$ be a twice continuously differentiable function with compact support, that is $u(x) = 0$ outside some compact set in $\mathbb{R}^n$. Then*

$$u(0) = \int_{\mathbb{R}^n} \Psi(x)\, \Delta u(x)\, dx$$

Proof. Find $R > 0$ such that $u(x) = 0$ for $|x| \geq R$. Notice that

$$\Psi(x)\, \Delta u(x)$$

is integrable in $B_0(R)$ (in the improper sense). Indeed, $\Delta u(x)$ is continuous, hence bounded. The only singular point for $\Psi(x)$ is the origin. But there it is integrable because $\ln|x|$ is integrable for $n = 2$ and $|x|^{-p}$ is integrable in $B_0(R)$ for $p < n - 1$ (apply the spherical Fubini theorem).

Therefore, removing a small ball with center at the origin, $V_\varepsilon = B_0(R) \backslash \overline{B_0(\varepsilon)}$, we obtain for the improper integral:

$$\int_{\mathbb{R}^n} \Psi\, \Delta u\, dx \equiv \int_{B_0(R)} \Psi\, \Delta u\, dx = \lim_{\varepsilon \to 0+} \int_{V_\varepsilon} \Psi\, \Delta u\, dx$$

We apply the argument given described in the Mean Value Theorem. Again we can assume that $n \geq 3$. Notice that $u = 0$ and $\partial_\nu u = 0$ on the boundary sphere $\partial B_0(R)$. Applying the Green identity and $\Delta\Psi = 0$, we find

$$\int_{V_\varepsilon} \Psi\, \Delta u\, dx = \int_{V_\varepsilon} (\Psi\, \Delta u - u\Delta\Psi)\, dx = -w(\varepsilon) \int_{\partial B_0(\varepsilon)} \partial_\nu u\, dS_x + \partial_r w(\varepsilon) \int_{\partial B_0(\varepsilon)} u\, dS_x$$

where $\Psi(x) = w(|x|)$ and

$$w(r) = -\frac{1}{(n-2)\omega_n r^{n-2}}.$$

Since $\partial_\nu u$ is bounded, $w(\varepsilon) \sim \varepsilon^{2-n}$ and $|\partial B_0(\varepsilon)| \sim \varepsilon^{n-1}$, we obtain

$$\lim_{\varepsilon \to 0+} w(\varepsilon) \int_{\partial B_0(\varepsilon)} \partial_\nu u \, dS_x = 0.$$

Next, noticing that $\partial_r w(\varepsilon) = \frac{1}{\omega_n \varepsilon^{n-1}}$, we find

$$\lim_{\varepsilon \to 0+} \int_{V_\varepsilon} \Psi \, \Delta u \, dx = \lim_{\varepsilon \to 0+} \partial_r w(\varepsilon) \int_{\partial B_0(\varepsilon)} u \, dS_x = \lim_{\varepsilon \to 0+} \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_0(\varepsilon)} u \, dS_x = u(0)$$

and the theorem is proved. ■

Corollary (Solution of the Poisson equation). *Let $f(x)$ be twice continuously differentiable, and let $f(x) = 0$ outside some compact set in $\mathbb{R}^n$. Define*

$$u(x) = \int_{\mathbb{R}^n} \Psi(x - y) f(y) dy$$

Then $u(x) \in C^2(\mathbb{R}^n)$ and $\Delta u = f$ in $\mathbb{R}^n$.

Appendix: the Dirac delta-function

Another equivalent formulation of the above characteristic property of the fundamental solution is

$$\Delta\Psi(x) = \delta_0,$$

where δ_0 is the Dirac delta. Indeed, we use the following naïve definition of δ_0 : the identity

$$\int_{\mathbb{R}^n} \delta_0(x) f(x) dx = f(0),$$

holds for any smooth function f having a compact support. There is no classic function which could satisfy the above identity, that is why the Dirac delta is called a generalized function.

Consider the three-dimensional fundamental solution:

$$u(x) \equiv \Psi(y - x).$$

Another way to write this is

$$u(0) \equiv \Psi(y) = \int_{\mathbb{R}^n} \Psi(y - x) \delta_0(x) \, dx = \int_{\mathbb{R}^n} \Psi(z) \delta_0(y - z) \, dz.$$

Comparing this with the characteristic property of the fundamental solution:

$$u(0) = \int_{\mathbb{R}^n} \Psi(z) \, \Delta u(z) \, dz$$

implies the desired formula $\Delta_z \Psi(y - z) = \delta_0(y - z)$.