Lecture 10: The Green function

We already know that that the Laplacian of the fundamental solution Ψ is 'almost' zero:

$$\Delta\Psi(x) = \delta_0.$$

This property, appropriately stated, yields (see Appendix for the proof) that

$$u(x) = \int_\Omega \Psi(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} \big(u(y)\,\partial_\nu\Psi(x-y) - \Psi(x-y)\;\partial_\nu u(y)\big)dS_y$$

If one replace Ψ by $\Phi(\mathrm{x}, y) = \Psi(x-y) + v(y)$ in the above formula, where $v(y)$ is a harmonic (continuous) in Ω function, then one finds (by virtue of the second Green identity) that

$$u(x) = \int_\Omega \Phi(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} \big(u(y)\,\partial_\nu\Phi(x-y) - \Phi(x-y)\;\partial_\nu u(y)\big)dS_y.$$

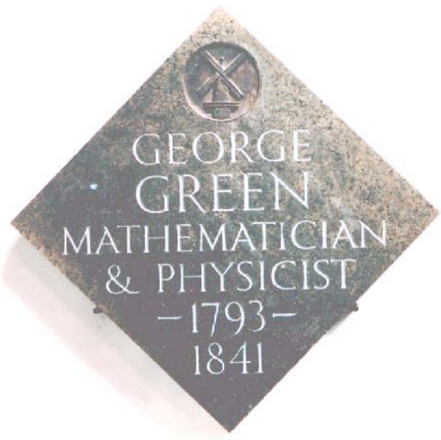
Now suppose that we are able to find the function $v(y)$ such that Φ vanishes on the boundary:

$$\Phi(x,y) \equiv \Psi(x-y) + v(y) = 0, \qquad y \in \partial\Omega.$$

Then for a harmonic function u the above integral identity takes the form:

$$u(x) = \int_{\partial\Omega} u(y)\,\partial_\nu\Phi(x-y)\,dS_y.$$

Thus we obtain the solution of the Dirichlet problem by using an effective integral representation.



It was George Green who was suggested first this idea as early as 1828. In his *Essay* (1828) he introduced the idea of potential function (what is also known today as a potential). The function Φ above is called the *Green function* and it is just a particular case of very general concept. Below we treat some basic properties of the Green function for the Dirichlet problem.

Definition. Given a domain Ω, the **Green function** for the Dirichlet problem, $G(x,y)$, is a harmonic (except for the diagonal $x = y$) function in each variable and such that

$$G(x,y) - \Psi(x-y) \text{ is continuous in } \Omega \text{ and } G(\cdot,y)|_{\partial\Omega} \equiv 0.$$

In other words, $G(x,y)$ is harmonic as a function of x and has singularity at $x=y$ of the same type as the fundamental solution. We rewrite the above integral representation as

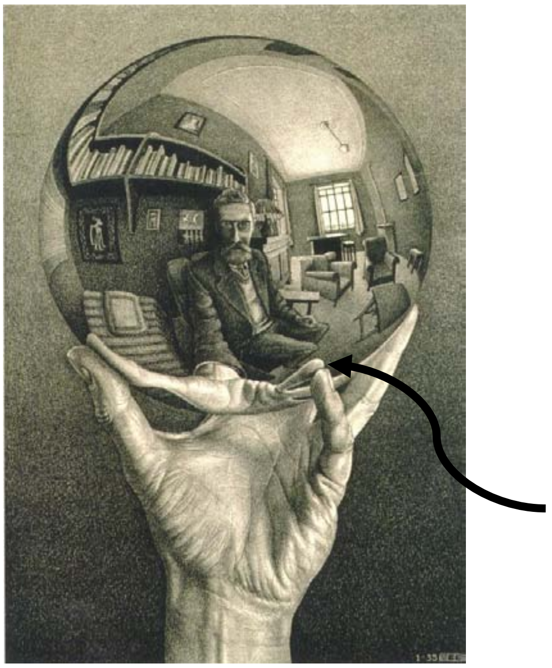
$$u(x) \equiv \int_{\partial\Omega} u(y)P(x,y)dS_y,$$

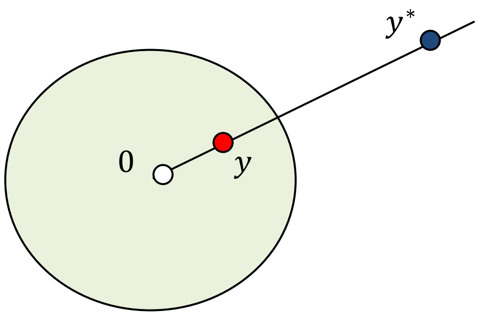
where $P(x,y)=\partial_{\nu_y}G(x,y)$ is called the **Poisson kernel** (of the domain Ω.)

Example: The Green function and the Poisson kernel for the ball $B_0(R)$. We shall use the reflection method which allows one to construct the `regularization' function $v(y)$ by using the rotational symmetry of the ball. Let $y \neq 0$ be any point in the ball and let

$$y^* = \frac{R^2 y}{|y|^2}$$

be the **inverse** of y with respect to the sphere $\partial B_0(R)$:





If $y \in \partial B_0(R)$ then $|y| = R$, hence $y^* = y$.

Magic mirror by *Mouritz Escher*

Then we find for a point $x \in \partial B_0(R)$ (i.e. $|x| = R$) that

$$|x-y^*|^2 = \langle x-y^*, x-y^*\rangle = |x|^2 - 2\langle x, y^*\rangle + |y^*|^2 = R^2 - \frac{2R^2}{|y|^2}\langle x,y\rangle + \frac{R^4}{|y|^2}$$

$$= (\text{since } |x| = R) = \frac{R^2}{|y|^2}(|y|^2 - 2\langle x,y\rangle + |x|^2) = \frac{R^2}{|y|^2}|y-x|^2.$$

That is

$$|x-y^*| = \frac{R}{|y|}|x-y|, \text{ for } x \in \partial B_0(R).$$

The latter identity indicates the way how to find the Green function. Indeed, let us consider the function (for dimension $n \geq 3$)

$$G(x,y) = \Psi(x-y) - \left(\frac{R}{|y|}\right)^{n-2} \Psi(x-y^*)$$

Then setting $c = -\frac{1}{\omega_n(n-2)}$ we find for $x \in \partial B_0(R)$:

$$G(x,y) = c\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|y|}\right)^{n-2} \cdot \frac{1}{|x-y^*|^{n-2}}\right) = c\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y|^{n-2}}\right) = 0,$$

hence $G(x,y)$ vanishes for x on the boundary sphere. It is clear also that the singularity of $G(\cdot,y)$ is located at y and that

$$v(x) \coloneqq G(x,y) - \Psi(x-y) = -\left(\frac{R}{|y|}\right)^{n-2} \Psi(x-y^*)$$

is a continuous harmonic function. By the definition, $G(x,y)$ is the Green function of the ball.

Now we find the Poisson kernel. We have for the unit normal $\nu_x = \frac{x}{R}$, hence

$$P(x,y) = \partial_{\nu_x} G(x,y) = \frac{1}{R}\langle \nabla G(x,y), x\rangle,$$

where the gradient is with respect to x. We have

$$\nabla\Psi(x-y) = -\frac{c(n-2)(x-y)}{|x-y|^n},$$

$$\nabla\left(\frac{R}{|y|}\right)^{n-2} \Psi(x-y^*) = -\left(\frac{R}{|y|}\right)^{n-2} \cdot \frac{c(n-2)(x-y^*)}{|x-y|^n}.$$

This yields (recall that $x \in \partial B_0(R)$)

$$P(x,y) = \frac{c(n-2)}{R}\langle\left(\frac{R}{|y|}\right)^{n-2} \frac{(x-y^*)}{|x-y^*|^n} - \frac{(x-y)}{|x-y|^n}, x\rangle =$$

$$= \frac{c(n-2)}{R}\langle\frac{|y|^2}{R^2} \cdot \frac{x-y^*}{|x-y|^n} - \frac{x-y}{|x-y|^n}, x\rangle = \frac{1}{R\omega_n|x-y|^n} \cdot |x|^2\left(1 - \frac{y^2}{R^2}\right).$$

But $|x| = R$, whence

$$P(x,y) = \frac{1}{R\omega_n|x-y|^n} \cdot (R^2 - |y|^2).$$

Thus we have proved

The Poisson formula. *The solution of the Dirichlet problem: $\Delta u(x) = 0$ in $B_0(R)$ and $u = g(x)$ on the boundary, is given by the formula*

$$u(y) = \frac{R^2 - |y|^2}{R\,\omega_n} \int_{|x|=R} \frac{g(x)}{|x-y|^n} dS_x$$

A similar argument shows that for a half space $\mathrm{H} = \{x : x_n > 0\}$ the Green function is

$$G(x,y) = \Psi(x-y) - \Psi(x-y^*), \qquad y^* = (y_1, \dots, y_{n-1}, -y_n),$$

and the Poisson kernel given by

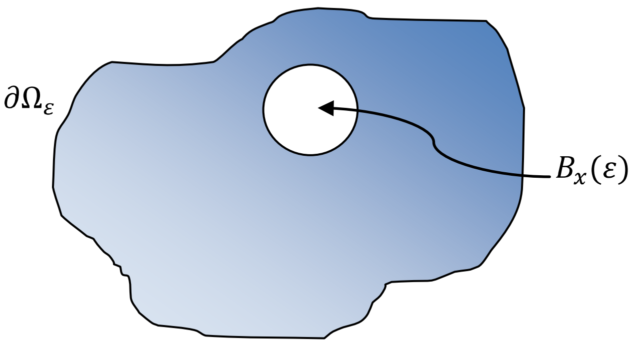
$$P(x,y) = \frac{2y_n}{\omega_n |x-y|^n}, \qquad x \in \partial\mathrm{H}.$$

Appendix

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $u(x) \in C^2(\bar{\Omega})$ (twice continuously differentiable up to the boundary). Then*

$$u(x) = \int_\Omega \Psi(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} \big(u(y)\,\partial_\nu \Psi(x-y) - \Psi(x-y)\,\partial_\nu u(y)\big) dS_y$$

Proof. Let $x \in \Omega$ and let $B_x(2\varepsilon) \subset \Omega$. Consider $\Omega_\varepsilon = \Omega \setminus \overline{B_x(\varepsilon)}$



Applying the second Green identity we find

$$\int_{\Omega_\varepsilon} \Psi(x-y)\,\Delta u(y)\,dy = \int_{\partial\Omega_\varepsilon} \big(u(y)\,\partial_\nu \Psi(x-y) - \Psi(x-y)\,\partial_\nu u(y)\big) dS_y$$

where $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_x(\varepsilon)$. Now let $\varepsilon \to 0$.

- Since Ψ is (improperly) integrable, the integral over Ω_ε converges to the integral over Ω as $\varepsilon \to 0$.

- The integral over sphere $\partial B_x(\varepsilon)$ is (for $n \geq 3$)

$$\varepsilon^{n-1} \int_{|t|=1} \big(u(x+\varepsilon t)\,\partial_\nu \Psi(\varepsilon t) - \Psi(\varepsilon t)\,\partial_\nu u(x+\varepsilon t)\big) dS_t =$$

$$= \varepsilon^{n-1} \int_{|t|=1} \left(\frac{u(x+\varepsilon t)}{\omega_n \varepsilon^{n-1}} - \frac{1}{(n-2)\varepsilon^{n-2}}\,\partial_\nu u(x+\varepsilon t) \right) dS_t$$

Hence it converges to $u(x)$ (since the latter integral converges to zero) and we obtain the required formula. ■