Lecture 11: The heat equation

Now we consider parabolic equations, that is equations with one characteristic (the characteristic discriminant is zero). The main model example is the *heat* or *diffusion* equation in $\mathbb{R}^n$

$$u_t = k\Delta u \equiv k\big(u_{x_1x_1} + \ldots + u_{x_nx_n}\big), \qquad x \in \Omega,\ \ t > 0.$$

We already discussed the derivation of the heat equation (see Lecture 4) and know that it governs the propagation (or diffusion) of the heat measured in terms of temperature $u(x,t)$ at the point x and time t. In contrast to the wave equation when the constant k must be positive (it is the squared speed of waves), here the sign of the constant k is non-essential, and we shall assume in what follows that $k = 1$ (notice that the new function $v(x,t) = u(x,t/k)$ satisfies the heat equation with $k = 1$).

It follows from the derivation of the heat equation that a reasonable initial condition is the distribution of the initial temperature, that is

$$u(x,0) = g(x)$$

and, may be some other boundary data like Dirichlet or Neumann boundary values describing possible obstacles or conditions for behavior of the temperature on the boundary $\partial\Omega$.

The eigenfunctions method

We try first to solve the heat equation subject to the most standard conditions:

$$\begin{aligned} &u_t = \Delta u,\ \ x \in \Omega,\ \ t > 0 \\ &u(x,0) = g(x) \quad \text{in } \Omega \qquad\qquad (1)\\ &u(x,t) = 0 \quad \text{on the boundary } \partial\Omega. \end{aligned}$$

In order to be consistent with the second condition we assume that $g = 0$ on the boundary and that g is of class C^2 inside the domain. Applying the separation of variables $u(x,t) = v(x)w(t)$, we find

$$w'(t) = Cw(t) \qquad (2)$$

$$\Delta v(x) = Cv(x) \qquad (3)$$

where C is some constant. In this notation, the second equation becomes the eigenvalue problem for the Laplace operator. Indeed, by the above boundary conditions above we have $v(x) = 0, x \in \partial\Omega$. Multiplying $\Delta v(x) = Cv(x)$ by v and integrating we obtain

$$\int_\Omega Cv^2\,dx = \int_\Omega v\Delta v\,dx = \text{by the } 1^{\text{st}} \text{ Green identity} = -\int_\Omega |\nabla v|^2\,dx,$$

which shows that the constant C is non-negative:

$$C = -\lambda^2, \qquad \lambda^2 = \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx}.$$

We consider this eigenvalue problem below in more details but now describe briefly the **strategy of solving the above boundary problem (1)**:

i. Eq. (3) has a family of linear independent solutions ϕ_k (*eigenfunctions*) parameterized by the *eigenvalues* $\lambda_k > 0$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and normalized such that $\int_\Omega \phi^2 \, dx = 1$

ii. Complete this family to an orthonormal basis in $L^2(\Omega)$ with the scalar product $\langle \phi, \psi \rangle = \int_\Omega \phi\psi \, dx$, that is

$$\int_\Omega \phi_m \phi_n \, dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta.

iii. Any (continuous) function $g(x)$ in $L^2(\Omega)$ can be written as the sum

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad a_n = \langle g, \phi_n \rangle \equiv \int_\Omega g(x)\phi_n(x) \, dx$$

and the series converges uniformly on $\overline{\Omega}$.

iv. The solution of (1) can be found then in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x)$$

Let us demonstrate this method by the following example.

Example 1. Solve

$$u_t = u_{xx}, \quad x \in [0,\pi], \quad t > 0$$

$$u(x,0) = (\sin x)^3 \qquad \text{in} \quad [0,\pi]$$

$$u(0,t) = u(\pi,t) = 0$$

Solution. The eigenvalue Dirichlet problem for the one-dimensional interval and the one-dimensional Laplacian (the second derivative) leads to the trigonometric family

$$\phi_1 = \sin x\,, \qquad \phi_2 = \sin 2x\,, \ldots$$

which is well-known to be an orthogonal system on $[0,\pi]$ with respect to the scalar product

$$\langle \phi, \psi \rangle = \frac{2}{\pi} \int_0^\pi \phi(x)\psi(x) dx.$$

Notice that $\lambda_k = k$. We find

$$(\sin x)^3 = \sin x \ (\sin x)^2 = \sin x \ (1 - \cos 2x) = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \equiv \frac{3}{4} \phi_1 - \frac{1}{4} \phi_2.$$

Hence the solution is found as

$$u(x,t) = \frac{3}{4} \phi_1 e^{-\lambda_1 t} - \frac{1}{4} \phi_2 e^{-\lambda_2 t} = \frac{3}{4} e^{-t} \sin x - \frac{1}{4} e^{-3t} \sin 3x.$$

Dirichlet eigenvalues and eigenfunctions of the Laplacian

The general problem on eigenvalues and eigenfunctions of the Laplacian operator requires a deep familiarity with functional analysis, integral equations theory and even operator theory. Here we give a short outline of the basic approaches.

The variational principle allows to reduce the original for finding of the eigenvalues Laplace equation to the variational problem: find the minimum of the functional

$$\phi \to V(\phi) = \frac{\int_\Omega |\nabla\phi|^2 \, dx}{\int_\Omega \phi^2 \, dx}, \qquad \phi = 0 \ \text{ for } \ x \in \partial\Omega.$$

It turns out that the minimum ϕ_1 does exist and it is exactly the first eigenvunction

$$\Delta\phi_1 = -\lambda_1^2\phi_1, \qquad \phi_1 = 0 \ \text{ for } \ x \in \partial\Omega,$$

where λ_1^2 is the smallest (non-zero) eigenvalue. The next step is to find the minimum of $V(\phi)$ on the subspace consisting of all functions with zero-boundary values and orthogonal to ϕ_1, etc. This variational technique (*Rayleigh quotient*) allows to construct the eigenfunctions and eigenvalues step by step.

Integral Equation Method allows to use the Green function of the domain Ω to define the following integral equation:

$$u(x) = -\lambda \int_\Omega G(x, y)u(y)dy.$$

Since the Green function is symmetric, one can solve the above equation similar to that for symmetric matrices:

$$\lambda x = Ax, \qquad A^T = A.$$

One needs a special technique, which generalizes the finite dimensional eigenvalue problem onto the infinite dimensional case.

Functional Analysis and operator theory methods recognize the Laplace operator as a self-adjoint operator on the (infinite dimensional) space consisting of square-integrable functions[1]

$$L^2(\Omega) = \{\phi: \int_\Omega \phi^2(x)dx < \infty\}$$

Indeed, applying the 2nd Green identity to any two smooth functions ϕ, ψ in $L^2(\Omega)$ having zero boundary values, we obtain

$$\langle\Delta\phi, \psi\rangle = \int_\Omega \psi\Delta\phi \, dx = \int_\Omega \phi\Delta\psi \, dx = \langle\phi, \Delta\psi\rangle \qquad \Rightarrow \qquad \Delta^* = \Delta.$$

[1] One has to use the Lebesgue integral in order to obtain a *complete* vector space.

Now we summarize the properties of the eigenvalues and eigenfunctions which follow from the general theory.

- The eigenvalues λ_n^2 form a countable set and $\lambda_n \to \infty$ as $n \to \infty$.
- For each λ_n there is only a finitely many linear independent eigenfunctions $\phi_n$with this eigenvalue.
- The first eigenvalue has multiplicity one.
- Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Example 2. We exemplify the above properties by eigenfunctions of the Laplacian in the rectangle

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a, 0 \leq y \leq b\}.$$

Then the eigenvalue problem can be solved by separation of variables method. Substitution of $u = v(x)w(y)$ for $\Delta u = -\lambda^2 u$ leads to the following funcitons

$$\phi_{nm}(x,y) = \sin\frac{\pi n x}{a}\sin\frac{\pi m y}{b}, \qquad n, m = 1,2, \ldots,$$

where the eigenvalues λ_{nm} are found easily to be

$$\lambda_{nm} = \pi\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}.$$

Observe that for some configurations of a and b there multiple eigenfunctions. For example, if $a = b$ then

$$\lambda_{5,5} = \lambda_{1,7} = \lambda_{7,1}.$$

But it easy to see that for any number $M > 0$ there are only finitely many eigenvalues satisfyng $\lambda_{nm} < M$.

In order to prove the orthogonality, notice that

$$\langle \phi_{nm}, \phi_{kl} \rangle = \int_0^a \sin\frac{\pi n x}{a}\sin\frac{\pi k x}{a}dx \cdot \int_0^b \sin\frac{\pi m y}{b}\sin\frac{\pi l y}{b}dy = \left(\frac{2}{\pi}\right)^2 ab\,\delta_{nk}\delta_{ml}.$$

The coefficients of the expansion of $g(x,y)$ with zero boundary values in the double series

$$g(x,y) = \sum_{n,m=1}^{\infty} A_{nm}\phi_{nm}(x,y)$$

is given by

$$A_{nm} = \frac{4}{ab}\int_0^a dx \int_0^b g(x,y)\sin\frac{\pi n x}{a}\sin\frac{\pi m y}{b}dy.$$

<u>The maximum principle</u>

Notice that the time independent solutions of the heat equation satisfy the Laplace equation. On the other hand, the behavior of series

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x)$$

makes it plausible to assume that the solution is stabilized 'at infinity', that is $u(x,+\infty)$ becomes a harmonic function. In order to make this observation more rigorous we start first with the maximum principle for the heat equation.

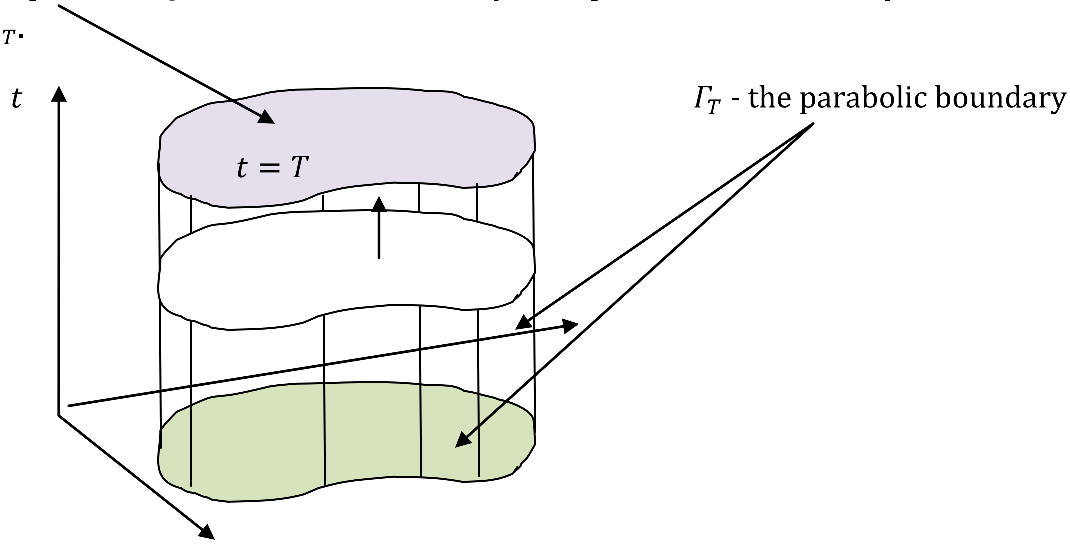
Consider the heat equation

$$u_t = \Delta u, \quad x \in \Omega, \quad t > 0$$

in the cylinder

$$\Omega_T = \Omega \times [0,T] = \{(x,t) : x \in \Omega, \qquad 0 < t < T\}.$$

It is reasonable to think of the `cover' $\Omega \times \{T\}$ as the *inner* boundary, while regard the remained *exterior* part of $\partial\Omega_T$ as the factual boundary. This part is also called the *parabolic* boundary of Ω_T.



Theorem (Weak maximum principle) Let u be a smooth function (twice continuously differentiable in x and continuously differentiable in t) in U and continuous up to the boundary. Let u satisfy the inequality

$$\Delta u \geq u_t, \qquad (x,y) \in \Omega_T.$$

Then u achieves its maximum on the parabolic boundary:

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

Proof. First let us assume that a stronger inequality holds: $\Delta u > u_t$ in Ω_T and consider a subcylinder Ω_τ, $\tau < T$.

Let (x_0, τ) be a maximum on $\overline{\Omega_\tau}$, where $x_0 \in \Omega$. Then $u_t(x_0, \tau) \geq 0$ and, moreover, $D^2 u(x_0, \tau) \leq 0$ (the matrix of second derivatives is non-positive definite). The latter implies $\Delta u(x_0, \tau) \leq 0$ (the trace of the matrix is non-positive). Hence we get

$$\Delta u(x_0, \tau) \leq 0 \leq u_t(x_0, \tau),$$

which contradicts our assumption. Similarly, if (x_0, τ_0) is an inner point, we obtain the $\nabla u(x_0, \tau_0) = 0$ and $\Delta u(x_0, \tau_0) \leq 0$.

Hence $\max_{(x,t) \in \overline{\Omega_\tau}} u(x,t) = \max_{(x,t) \in \Gamma_\tau} u(x,t)$, and by continuity we obtain the desired inequality.

Now we consider the original inequality and introduce an auxiliary function $v = u - kt$, $k > 0$. Clearly $\Delta v - v_t = \Delta u - u_t + k > 0$. We apply the previous argument:

$$\max_{\overline{\Omega_T}} v = \max_{\Gamma_T} v$$

Hence

$$\max_{\overline{\Omega_T}} u = \max_{\overline{\Omega_T}} (v + t) \leq \max_{\overline{\Omega_T}} v + kT = \max_{\Gamma_T} v + kT \leq \max_{\Gamma_T} u + kT.$$

Letting $k \to 0$ we arrive at the required inequality. ■

Corollary (Uniqueness). If u, v as in theorem above are solutions to

$$u_t = \Delta u + f(x,t), \quad x \in \Omega, \quad t > 0$$
$$u(x,0) = g(x) \quad \text{in } \Omega \tag{1}$$
$$u(x,t) = h(x,t) \quad x \in \partial\Omega, \quad t > 0,$$

then $u \equiv v$.