Lecture 12: The heat equation

The fundamental solution

There is no a radial symmetric solution of the heat equation as in the case with the Laplace equation. Instead, we show that the function (the heat kernel)

$$u(x, t) = \frac{1}{t^\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \ t > 0.$$  

which depends symmetrically on $x$ is a solution of the heat equation. Indeed,

$$u_t = -\frac{n}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{n}{2} t^{-\frac{n}{2}} \frac{e^{-\frac{|x|^2}{4t}} |x|^2}{4t^2} = t^{-\frac{n}{2}-1} \frac{e^{-\frac{|x|^2}{4t}}}{2} \left(\frac{|x|^2}{2t} - n\right)$$

and

$$\Delta e^{-\frac{|x|^2}{4t}} = -\frac{1}{2t} \text{div} \left(\Delta e^{-\frac{|x|^2}{4t}} x\right) = -\frac{e^{-\frac{|x|^2}{4t}}}{2t} \left(n - \frac{1}{2t} |x|^2\right)$$

Hence $\Delta u - u_t = 0$.

The significance of this function for the heat equation theory is seen from the following property. We illustrate this by the two-dimensional case. First we modify slightly our solution and define the new function $\psi: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$\psi = \begin{cases} 
\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\
0 & t \leq 0
\end{cases}$$

This function is called the fundamental solution of the heat equation in $\mathbb{R}^2$.

**Theorem.** The function $\psi$ is locally integrable in $\mathbb{R}^2$, that is it is integrable on any bounded open set. Moreover, for any function $\phi(x, t) \in C^2(\mathbb{R}^2)$ having compact support the following identity holds

$$\int_{\mathbb{R}^2} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2}\right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \ dx = -\phi(0,0).$$

**Proof.** Observe that $\psi$ is locally integrable. Indeed, we have

$$0 \leq \psi \leq \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \leq \frac{1}{2\sqrt{\pi t}}.$$

The latter square root function is integrable in any squares (why?)

$$Q = \{|x| \leq M, |t| \leq M\}, \ M > 0$$
Hence $\psi$ is locally integrable in $\mathbb{R}^2$.

Let $\phi(x, t)$ be any twice differentiable function with compact support in $\mathbb{R}^2$. We chose $M$ such that the support of $\phi$ is contained in the strip $|x| < M$. We know that

$$u = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

is a solution of the heat equation for $t > 0$. Consider the following integral

$$I(\phi) = \int_{\mathbb{R}^2} \left( \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} dx = \lim_{\varepsilon \to 0} \int_{Q_\varepsilon} \left( \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) u(x, t) dx = \lim_{\varepsilon \to 0} I_\varepsilon,$$

where $Q_\varepsilon = \{|x| \leq M, t > \varepsilon\}$. We find the last integral $I_\varepsilon$ by splitting it into two parts:

$$I_1 = \int_\varepsilon^M dt \int_{-M}^M \phi_{xx} u \, dx$$

and

$$I_2 = \int_\varepsilon^M dt \int_{-M}^M \phi_t u \, dx.$$ 

By integrating the inner integral in $I_1$ by parts two times (recall that $\phi(\pm M, t) = \phi(x, \pm M) = 0$, hence the boundary values are equal to zero) we obtain

$$\int_{-M}^M u \phi_{xx} \, dx = \int_{-M}^M u \, d\phi_x = \phi_x \big|_{x=-M}^{x=M} - \int_{-M}^M u_x \phi_x \, dx = \int_{-M}^M u_{xx} \phi \, dx$$

Thus

$$I_1 = \int_\varepsilon^M dt \int_{-M}^M u_{xx} \phi \, dx$$

Similarly,

$$I_2 = \int_\varepsilon^M dt \int_{-M}^M u \phi_t \, dx = \int_{-M}^M dx \int_\varepsilon^M u \, d\phi = \int_{-M}^M dx \left( u\phi \big|_{t=\varepsilon}^M - \int_\varepsilon^M u_t \phi \, dt \right) =$$
In summary,

\[ I(\epsilon) = l_1 + l_2 = -\int_{-M}^M h(x, \epsilon) \phi(x, \epsilon) dx - \int_{-M}^M dx \int_{\epsilon}^{M} (u_t - u_{xx}) \phi \, dt = \]

\[ = -\int_{-M}^M h(x, \epsilon) \phi(x, \epsilon) dx = -\int_{-\infty}^{\infty} h(x, \epsilon) \phi(x, \epsilon) dx \]

(recall that \( u_t - u_{xx} = 0 \)). Setting \( x = 2\sqrt{\epsilon}y \) we find

\[ I(\epsilon) = -\int_{-\infty}^{\infty} \phi(x, \epsilon) \frac{x^2}{2\sqrt{\pi\epsilon}} \, dx = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \phi(2\sqrt{\epsilon}y, \epsilon) \, dy \]

and since, \( \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi} \), we obtain

\[ \lim_{\epsilon \to 0} I(\epsilon) = -\phi(0,0), \]

This finally yields the desired formula and the theorem is proved. ■

The conjugate operator

Rewrite our formula as

\[ I(\phi) \equiv -\int_{\mathbb{R}^2} \left( \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \psi \, dx = \phi(0,0) = \delta_0(\phi), \]

where \( \delta_0 \) as usually denotes the Dirac delta. Denote by \( L \) the heat operator \( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \). Then the operator \( L^* = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \) under the integral is called the conjugate to \( L \). This terminology is seen from the following observation. If both functions \( \phi \) and \( \psi \) are smooth with compact supports then

\[ \langle \phi, L\psi \rangle \equiv \int_{\mathbb{R}^2} \phi L(\psi) \, dx = \int_{\mathbb{R}^2} \phi(\psi_t - \psi_{xx}) \, dx = \text{by integrating by parts} \]

\[ = \int_{\mathbb{R}^2} -\phi_t \psi \, dx - \int_{\mathbb{R}^2} \phi_{xx} \psi \, dx = \langle L^* \phi, \psi \rangle \]

Hence we interpret \( L^* \) as the conjugate operator with respect to the scalar product. Moreover,

\[ \langle \phi, L\psi \rangle = \langle L^* \phi, \psi \rangle = \int_{\mathbb{R}^2} L^*(\phi) \psi = \delta_0(\phi), \]

which is the characteristic property of a fundamental solution.
**Definition.** The function

\[\psi(x, t) = \begin{cases} \frac{1}{\pi} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}\]

is called the fundamental solution of the heat equation \(u_t - u_{xx} = 0\).

**The pure initial value problem**

\[
u_t = \Delta u, \quad x \in \mathbb{R}^n, \quad t > 0
\]

\[
u(x, 0) = g(x), \quad x \in \mathbb{R}^n
\]

**Theorem.** If \(g\) is bounded and continuous in the whole \(\mathbb{R}^n\) then

\[
u(x, t) = \int_{\mathbb{R}^n} \psi(x - y, t) g(y) \, dy = \frac{1}{\pi} \int_{\mathbb{R}^n} e^{-\frac{|x - y|^2}{4t}} g(y) \, dy
\]

is the solution of the above pure initial value problem: \(\nu\) is a \(C^\infty\)-function in \(\mathbb{R}^n \times (0, +\infty)\) and satisfies \(\nu_t = \Delta \nu\) there, and \(\nu\) continuously extended in the closed half-space \(\mathbb{R}^n \times [0, +\infty)\) such that \(\nu(x, 0) = g(x)\).

**Outline of the proof.**

- The integral is well defined for any fixed \(x \in \mathbb{R}^n\) and \(t > 0\) since \(g\) is bounded and the exponential is absolutely integrable in \(\mathbb{R}^n\).
- The fact that \(\nu\) is a \(C^\infty\)-function in \(\mathbb{R}^n \times (0, +\infty)\) follows from the standard theorems on differentiating of (improper) integral with respect to parameter.
- Differentiating with respect to \(x\) and \(t\) one proves that \(\nu\) is a solution of the heat equation.
- Prove that \(\int_{\mathbb{R}^n} \psi(x - y, t) g(y) \, dy = 1\) for any \(t > 0\) (Hint: use the scaling argument and the Gauss formula \(\int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi}\))
- Verify that for any \(\delta > 0\)

\[
\lim_{t \to +0} \int_{|x - y| > \delta} \psi(x - y, t) \, dy = 0
\]

uniformly for \(x \in \mathbb{R}^n\)
- Prove finally that \(\nu(x, t) \to g(x)\) as \(t \to 0^+\).