Lecture 12: The heat equation

<u>The fundamental solution</u>

There is no a radial symmetric solution of the heat equation as in the case with the Laplace equation. Instead, we show that the function (the **heat kernel**)

$$u(x,t) = \frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbb{R}^n, t > 0.$$

which depends symmetrically on x is a solution of the heat equation. Indeed,

$$u_t = -\frac{n}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} |x|^2}{4t^2} = t^{-\frac{n}{2}-1} \frac{e^{-\frac{|x|^2}{4t}}}{2} \left(\frac{|x|^2}{2t} - n \right)$$

and

$$\Delta e^{-\frac{|x|^2}{4t}} = -\frac{1}{2t} \mathrm{div} \left(\Delta e^{-\frac{|x|^2}{4t}} x \right) = -\frac{e^{-\frac{|x|^2}{4t}}}{2t} \left(n - \frac{1}{2t} |x|^2 \right)$$

Hence $\Delta u - u_t = 0$.

The significance of this function for the heat equation theory is seen from the following property. We illustrate this by the two-dimensional case. First we modify slightly our solution and define the new function $\psi: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$\psi = \begin{cases} \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0 & t \le 0 \end{cases}$$

This function is called the **fundamental solution** of the heat equation in $\mathbb{R}^2$.

Theorem. *The function ψ is locally integrable in $\mathbb{R}^2$, that is it is integrable on any bounded open set. Moreover, for any function $\phi(x,t) \in C^2(\mathbb{R}^2)$ having compact support the following identity holds*

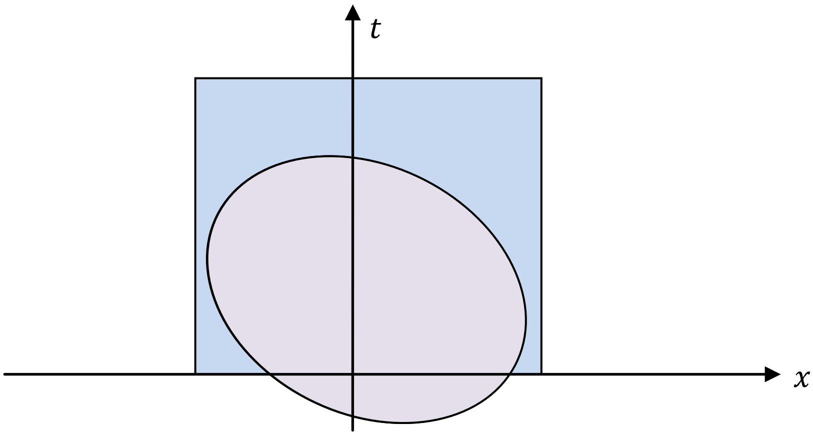
$$\int_{\mathbb{R}^2} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \, dx = -\phi(0,0).$$

Proof. Observe that ψ is locally integrable. Indeed, we have

$$0 \le \psi \le \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \le \frac{1}{2\sqrt{\pi t}}.$$

The latter square root function is integrable in any squares (why?)

$$Q = \{|x| \le M, |t| \le M\}, \ M > 0$$



Hence ψ is locally integrable in $\mathbb{R}^2$.

Let $\phi(x,t)$ be any twice differentiable function with compact support in $\mathbb{R}^2$. We chose Msuch that the support of ϕ is contained in the strip $|x| < M$. We know that

$$u = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

is a solution of the heat equation for $t > 0$. Consider the following integral

$$I(\phi) = \int_{\mathbb{R}^2} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \, dx = \lim_{\epsilon \to 0} \int_{Q_\epsilon} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) u(x,t) \, dx = \lim_{\epsilon \to 0} I_\epsilon \,,$$

where $Q_\epsilon = \{|x| \le M, t > \epsilon\}$. We find the last integral I_ϵ by splitting it into two parts:

$$I_1 = \int_\epsilon^M dt \int_{-M}^M \phi_{xx} \, u \, dx$$

and

$$I_2 = \int_\epsilon^M dt \int_{-M}^M \phi_t \, u \, dx.$$

By integrating the inner integral in I_1 by parts two times (recall that $\phi(\pm M, t) = \phi(x, \pm M) = 0$, hence the boundary values are equal to zero) we obtain

$$\int_{-M}^M u \;\phi_{xx} dx = \int_{-M}^M u \; d\phi_x = \phi_x \, u|_{x=-M}^{x=M} - \int_{-M}^M u_x \;\phi_x \; dx = \int_{-M}^M u_{xx} \;\phi \; dx$$

Thus

$$I_1 = \int_\epsilon^M dt \int_{-M}^M u_{xx} \phi \;\, dx$$

Similarly,

$$I_2 = \int_\epsilon^M dt \int_{-M}^M u \;\phi_t dx = \int_{-M}^M dx \int_\epsilon^M u \, d\phi \;= \int_{-M}^M dx \left(u\phi|_{t=\epsilon}^{t=M} - \int_\epsilon^M u_t \phi \, dt \right) =$$

$$= -\int_{-M}^{M} u(x,\epsilon)\phi(x,\epsilon)dx - \int_{-M}^{M} dx \int_{\epsilon}^{M} u_t \, \phi \, dt$$

In summary,

$$I(\epsilon) = I_1 + I_2 = -\int_{-M}^{M} h(x,\epsilon)\phi(x,\epsilon)dx - \int_{-M}^{M} dx \int_{\epsilon}^{M} (u_t - u_{xx})\phi \, dt =$$

$$= -\int_{-M}^{M} h(x,\epsilon)\phi(x,\epsilon)dx = -\int_{-\infty}^{\infty} h(x,\epsilon)\phi(x,\epsilon)dx$$

(recall that $u_t - u_{xx} = 0$). Setting $x = 2\sqrt{\epsilon}y$ we find

$$I(\epsilon) = -\int_{-\infty}^{\infty} \frac{\phi(x,\epsilon)}{2\sqrt{\pi\epsilon}} e^{-\frac{x^2}{4\epsilon}} dx = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} \phi\big(2\sqrt{\epsilon}y, \epsilon\big) dy$$

and since, $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$, we obtain

$$\lim_{\epsilon \to 0} I(\epsilon) = -\phi(0,0),$$

This finally yields the desired formula and the theorem is proved. ■

<u>The conjugate operator</u>

Rewrite our formula as

$$I(\phi) \equiv -\int_{\mathbb{R}^2} \left(\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} \right) \psi \, dx = \phi(0,0) = \delta_0(\phi),$$

where δ_0 as usually denotes the Dirac delta. Denote by L the heat operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$. Then the operator $L^* = -\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ under the integral is called the **conjugate** to L. This terminology is seen from the following observation. If both functions ϕ and ψ are smooth with compact supports then

$$\langle \phi, L\psi \rangle \equiv \int_{\mathbb{R}^2} \phi L(\psi) dx = \int_{\mathbb{R}^2} \phi(\psi_t - \psi_{xx}) dx = \text{by integrating by parts}$$

$$= \int_{\mathbb{R}^2} -\phi_t \psi \, dx - \int_{\mathbb{R}^2} \phi_{xx} \psi \, dx = \langle L^* \phi, \psi \rangle$$

Hence we interpret L^* as the conjugate operator with respect to the scalar product. Moreover,

$$\langle \phi, L\psi \rangle = \langle L^* \phi, \psi \rangle = \int_{\mathbb{R}^2} L^*(\phi) \, \psi = \delta_0(\phi),$$

which is the characteristic property of a fundamental solution.

Definition. The function

$$\psi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \le 0 \end{cases}$$

is called the fundamental solution of the heat equation $u_t - u_{xx} = 0$.

<u>The pure initial value problem</u>

$$u_t = \Delta u, \qquad x \in \mathbb{R}^n, \ \ t > 0$$

$$u(x,0) = g(x), \quad x \in \mathbb{R}^n$$

Theorem. *If g is bounded and continuous in the whole $\mathbb{R}^n$ then*

$$u(x,t) = \int_{\mathbb{R}^n} \psi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

is the solution of the above pure initial value problem: u is a C^∞-function in $\mathbb{R}^n \times (0,+\infty)$ and satisfies $u_t = \Delta u$ there, and u continuously extended in the closed half-space $\mathbb{R}^n \times [0,+\infty)$ such that $u(x,0) = g(x)$.

Outline of the proof.

- The integral is well defined for any fixed $x \in \mathbb{R}^n$ and $t > 0$ since g is bounded and the exponential is absolutely integrable in $\mathbb{R}^n$.
- The fact that u is a C^∞-function in $\mathbb{R}^n \times (0,+\infty)$ follows from the standard theorems on differentiating of (improper) integral with respect to parameter.
- Differentiating with respect to x and t one proves that u is a solution of the heat equation.
- Prove that $\int_{\mathbb{R}^n} \psi(x-y,t) g(y)\, dy = 1$ for any $t > 0$ (Hint: use the scaling argument and the Gauss formula $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$)
- Verify that for any $\delta > 0$

  $$\lim_{t \to +0} \int_{|x-y|>\delta} \psi(x-y,t)\, dy = 0$$

  uniformly for $x \in \mathbb{R}^n$
- Prove finally that $u(x,t) \to g(x)$ as $t \to 0+$.