Solutions to "Exercise problems (2009-02-10)"

Problem 1. Find a first-order differential equation having the following solutions:

a) $u = xy + f(x-y)$
b) $u = x + y + f(xy)$
c) $u = ax^2 + xy + ay^2$

Here f is an arbitrary function and a is an arbitrary constant.

Solution (1a) We have

$$u_x = y + f'(x-y), \qquad u_y = x - f'(x-y),$$

hence $u_x + u_y = y + x$.

By using a similar argument we find

Solution (1b) **:** $xu_x - yu_y = x - y$.

Solution (1c) **:** $yu_x - xu_y = y^2 - x^2$ (the answer is non-unique)

Problem 2. Find the general solution of the following homogeneous equations and draw the characteristic lines:

a) $u_x + yu_y = 0$,
b) $xu'_x + yu'_y + zu'_z = 0, \quad u = u(x,y,z)$

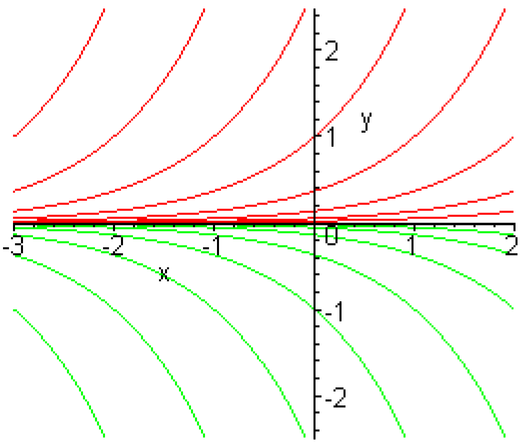
*Solution 2a***.** We have

$$\frac{dx}{1} = \frac{dy}{y} = \frac{du}{0}$$

Hence $u = const$ along characteristics. Moreover,

$$\frac{dy}{dx} = y \qquad \Rightarrow \qquad y = Ce^x,$$

so that we have for characteristics the following picture for the characteristic family:



Then the general solution is $u = F(C) = F(ye^{-x})$.

(Notice that, in particular, function $F(x - \ln y)$ is also a general solution.)

Solution 2b. The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{0}$$

Solving this system by usual methods we get

$$x = C_1 e^t, \qquad y = C_2 e^t, \qquad z = C_3 e^t, \qquad u = C_4.$$

Hence $\frac{x}{y}$ and $\frac{z}{y}$ are constant along characteristics. Since the Jacobian of these two functions is 2:

$$\operatorname{rank}\begin{pmatrix} \frac{1}{y} & -\frac{x}{y^2} & 0 \\ 0 & -\frac{z}{y^2} & \frac{1}{y} \end{pmatrix} = 2,$$

we can chose the general solution in the following form

$$u = F\left(\frac{x}{y}, \frac{z}{y}\right),$$

where F is an arbitrary function of two variables.

Problem 3. Find the general solution of the following equations:

a) $yu_x - xu_y = 1,$
b) $x^2 u_x + y^2 u_y = (x+y)u.$

Solution 3a. The characteristic equations are

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{du}{1}.$$

The first relation amounts to $d(x^2+y^2) = 0$, that is $x^2 + y^2 = C_1$ is the first constant. Now rewrite the above relations as

$$dx = ydu, \qquad dy = -xdu.$$

We find then $ydx - xdy = (x^2+y^2)du$, hence

$$d\arctan\frac{x}{y} \equiv \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{dx}{y} = \frac{ydx - xdy}{x^2+y^2} = du \qquad \Rightarrow \qquad u - \arctan\frac{x}{y} = C_2.$$

We have the general solution

$$f\left(u - \arctan\frac{x}{y}, x^2 + y^2\right) = 0.$$

Solution 3b. The characteristic equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{x+y}$$

yield $d\left(\frac{1}{x} - \frac{1}{y}\right) = 0$, whence $\frac{1}{x} - \frac{1}{y} = C_1$. Similarly, we can write

$$\frac{dx}{x} = \frac{xdu}{x+y}, \qquad \frac{dy}{y} = \frac{ydu}{x+y} \qquad \Rightarrow \qquad \frac{dx}{x} + \frac{dy}{y} = du,$$

that is $u - \ln xy = C_2$. Hence the general solution is $F(u - \ln xy, \frac{1}{x} - \frac{1}{y}) = 0$.

Problem 4. Solve the following Cauchy problems by method of characteristics:

a) $yu_x + xu_y = 0, \quad u(0,y) = y^2$
b) $3u_x + 2u_y = 0, \quad u(x,0) = \sin x$
c) $xu_x + yu_y = u + 1, \quad u(x,x^2) = x^2$

Solution 4a. The characteristic equations and the initial conditions are

$$\begin{cases} \dot{x} = y \\ \dot{y} = x\,, \\ \dot{u} = 0 \end{cases} \qquad \Gamma: \begin{cases} x_0 = 0 \\ y_0 = s \\ u_0 = s^2 \end{cases}$$

We have $\frac{d}{dt}(x \pm y) = (y \pm x)$, hence $x + y = C_1 e^t$ and $x - y = C_2 e^{-t}$. From the initial conditions we find

$$C_1 = 0 + s = s, \qquad C_2 = 0 - s = -s, \qquad u = u_0 = s^2.$$

Thus $x + y = se^t, x - y = -se^{-t}$, and it follows that $x^2 - y^2 = -s^2 = -u$. We have

$$u = y^2 - x^2.$$

Solution 4b. The characteristic equations and the initial conditions are

$$\begin{cases} \dot{x} = 3 \\ \dot{y} = 2\,, \\ \dot{u} = 0 \end{cases} \qquad \Gamma: \begin{cases} x_0 = s \\ y_0 = 0 \\ u_0 = \sin s \end{cases}$$

We have $x = 3t + x_0 = 3t + s$ and $y = 2t + y_0 = 2t$. Moreover, $u = u_0 = \sin s$. These equations imply

$$s = x - 3t = x - \frac{3}{2}y \quad \Rightarrow \quad u = \sin s = \sin\left(x - \frac{3}{2}y\right).$$

Solution 4c. The characteristic equations and the initial conditions are

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \\ \dot{u} = u + 1 \end{cases}, \qquad \Gamma: \begin{cases} x_0 = s \\ y_0 = s^2, \\ u_0 = s^2 \end{cases}$$

that is $x = C_1 e^t, y = C_2 e^t$ and $u = -1 + C_3 e^t$. From the initial conditions we find the unknown constants:

$$C_1 = x(0) = x_0 = s, \qquad C_2 = y(0) = s^2, \qquad C_3 = u(0) + 1 = 1 + s^2.$$

Hence $x = se^t$ and $y = s^2 e^t$,

$$u = -1 + (1 + s^2)e^t = -1 + e^t + s^2 e^t = -1 + e^t + y = -1 + \frac{x^2}{y} + y,$$

where

$$\frac{x^2}{y} = \frac{(se^t)^2}{s^2 e^t} = e^t.$$

Thus $u = -1 + \frac{x^2}{y} + y$.

Problem 5. Solve the following Cauchy problems by method of Lagrange:

a) $u_x + uu_y = y, \qquad u(0,y) = 1,$
b) $u_x + (x+y)u_y = 1, \quad u(x,-x) = 0.$

Solution 5a. We have

$$\frac{dx}{1} = \frac{dy}{u} = \frac{du}{y}, \qquad x_0 = 0,\ \ y_0 = s,\ \ u_0 = 1,$$

whence $d(u^2 - y^2) = 0$, which, by virtue of the initial condition yields

$$u^2 - y^2 = C_1 = 1 - s^2.$$

Furthermore, rewriting the system as

$$dy = udx, \qquad du = ydx,$$

and summing the equations we find $d(y+u) = (y+u)dx$, that is

$$d\ln(y+u) = dx.$$

We have similarly,

$$\ln(y+u) - x = C_2 = \ln(y_0 + u_0) - x_0 = \ln(1+s).$$

The relation between the two constants is easily found as follows:

$$s = e^{C_2} - 1 \quad \Rightarrow \quad C_1 = 1 - s^2 = 1 - (e^{C_2} - 1)^2 = 2e^{C_2} - e^{2C_2}.$$

Thus $C_1 = 2e^{C_2} - e^{2C_2}$ which gives

$$u^2 - y^2 = 2e^{\ln(y+u)-x} - e^{2(\ln(y+u)-x)} = 2(y+u)e^{-x} - (y+u)^2 e^{-2x},$$

and after cancellation of the common factor $(y+u)$:

$$u - y = 2e^{-x} - (y+u)e^{-2x}.$$

Thus we finally find that

$$u = \frac{1 + y\sinh x}{\cosh x}$$

Solution 5b. We have

$$\frac{dx}{1} = \frac{dy}{x+y} = \frac{du}{1}, \qquad x_0 = s,\ \ y_0 = -s,\ \ u_0 = 0.$$

The first constant is $x - u = C_1 = s - 0 = s$. In order to construct another constant we rewrite the first relation of the system as

$$\frac{dy}{dx} = x + y,$$

then this linear equation have the following solution:

$$y = -1 - x + C_2 e^x.$$

The initial condition yields

$$C_2 = (y + x + 1)e^{-x} = (y_0 + x_0 + 1)e^{-x_0} = 1 \cdot e^{-s} = e^{-s}.$$

Combining the relations for C_1 and C_2 we find that $C_2 = e^{-C_1}$, that is

$$(y + x + 1)e^{-x} = e^{u-x}.$$

Simplifying the obtained equation we find the solution

$$u = \ln(1 + x + y).$$

Problem 6. Solve the Cauchy problem

$$(1 + x^2)u'_x + 2xy\, u'_y = 0, \quad u(x, x + x^3) = h(x)$$

where $h(x)$ is some function.

Solution. We have the initial conditions $x_0 = s, y_0 = s + s^3, u_0 = h(s)$ and the characteristic equations

$$\dot{x} = 1 + x^2, \qquad \dot{y} = 2xy, \qquad u = u_0 = h(s).$$

From the first two equations we find

$$\frac{dy}{y} = \frac{2xdx}{1 + x^2} \quad \Rightarrow \quad \ln y - \ln(1 + x^2) = C_1 = \ln y_0 - \ln(1 + x_0^2) = \ln s,$$

hence $\frac{y}{1+x^2} = s$. Combining with $u = h(s)$ we find

$$u = h\left(\frac{y}{1 + x^2}\right).$$

Problem 7. Solve the initial-value problems for non-linear equations

a) $xu'_x + yu'_y + u'_x u'_y = u, \quad u(x, 0) = x^2$
b) $u_x = u_y^2, \quad u(0, y) = \frac{y^2}{2},$

by both characteristic method and the method of envelopes (try first affine solutions).

Solution 7a (Method of characteristics). Write our equation in pq-notation as

$$F \coloneqq xp + yq + pq - u = 0.$$

Then the initial conditions for the old variables are

$$x_0 = s, \qquad y_0 = 0, \qquad u_0 = s^2$$

and for the new variables p and q we have the following system

$$x_0\, p_0 + y_0 q_0 + p_0 q_0 - u_0 = 0,$$
$$u'_0 = x'_0 p_0 + y'_0 q_0.$$

The second equation amounts to $2s = 1 \cdot p_0$, hence $p_0 = 2s$ and q_0 is found from the first eq.:

$$q_0 = -\frac{s}{2}.$$

Now we write the characteristic equations:

$$\dot{x} = F_p = x + q$$

$$\dot{y} = F_q = y + p$$

$$\dot{u} = pF_p + qF_q = xp + yq + 2pq = u + pq$$

$$\dot{p} = -F_x - pF_u = -p + p = 0$$

$$\dot{q} = -F_y - qF_u = -q + q = 0.$$

The latter equations imply that $p = p_0 = 2s$ and $q = q_0 = -\frac{s}{2}$.

Moreover, the first equation gives

$$x = -q_0 + C_1 e^t = \frac{s}{2} + C_1 e^t,$$

and substituting the initial conditions yields:

$$x_0 = \frac{s}{2} + C_1 \quad \Rightarrow \quad C_1 = \frac{s}{2} \quad \Rightarrow \quad x = \frac{s}{2}(1 + e^t).$$

Similarly we find

$$y = 2s(e^t - 1).$$

The remaining equation $\dot{u} = u + pq = u + p_0 q_0 = u - s^2$ has the following solution: $u = C_2 e^t + s^2$. The initial condition yields $C_2 = 0$, hence $u = s^2$. Eliminating t from equations for x and y we find

$$s = x - \frac{y}{4}$$

that is

$$u = s^2 = \left(x - \frac{y}{4}\right)^2.$$

Solution 7a (Method of envelopes). We try first the affine solution $v = a + bx + cy + dxy$. After substitution

$$x(b + dy) + y(c + dx) + (b + dy)(c + dx) = a + bx + cy + dxy$$

and equating the coefficients of $1, x, y, xy$ to zero we obtain the system

$$a = bc, \qquad b(1 + d) = b, \qquad c(1 + d) = c, \qquad d(d + 1) = 0,$$

which yields a non-trivial solution $a = bc$ and $d = 0$. Hence the trial affine solution is

$$v = bc + bx + cy.$$

Now we take $c = kb$, where k will be chosen later:

$$v(x, y; b) = kb^2 + bx + kby$$

Then the envelope equation is

$$\frac{\partial}{\partial b} v(x, y; b) = 0$$

that is $2bk + x + ky = 0$, hence $b = -\frac{x+ky}{2k}$.

Substitution the found b into $v(x, y; b)$ gives

$$v(x, y; b) = k\left(-\frac{x + yk}{2k}\right)^2 - \frac{x + yk}{2k}(x + ky) = \frac{1}{4k} \cdot (x + ky)^2$$

Finally, verify the initial condition $u(x,0) = x^2$:

$$u(x,0) = \frac{1}{4k} \cdot x^2 = x^2 \qquad \Rightarrow \qquad 4k = 1$$

Hence $k = \frac{1}{4}$ and it follows that $u = \frac{1}{4k} \cdot (x + ky)^2 = \left(x + \frac{y}{4}\right)^2$.

Solution 7b (Method of characteristics). In this case we have

$$F \coloneqq p - q^2 = 0$$

and the initial conditions are $x_0 = 0$, $y_0 = s$ and $u_0 = \frac{s^2}{2}$. We find the additional initial conditions first:

$$p_0 - q_0^2 = 0,$$

$$u_0' = x_0' p_0 + y_0' q_0 \qquad \Rightarrow \qquad s = q_0.$$

Hence $p_0 = s^2$.

The characteristic equations

$$\dot{x} = F_p = 1$$

$$\dot{y} = F_q = -2q$$

$$\dot{u} = pF_p + qF_q = p - 2q^2 = -p$$

$$\dot{p} = -F_x - pF_u = 0$$

$$\dot{q} = -F_y - qF_u = 0$$

show that $p = p_0 = s^2$ and $q = q_0 = s$. This yields

$$x = t + x_0 = t, \qquad y = -2q_0 t + y_0 = -2st + s$$

and

$$u = -p_0 t + u_0 = -s^2 t + \frac{s^2}{2} = \frac{s^2}{2}(1 - 2t).$$

Eliminating s: $s = \frac{y}{1-2x}$ we find

$$u = \frac{\left(\frac{y}{1-2x}\right)^2 (1 - 2x)}{2} = \frac{y^2}{2(1-2x)}.$$

Solution 7a (Method of envelopes). Given in lecture notes.

Problem 8. Find the solution to the Cauchy problem: $x{u_x'}^2 + y u_y' = 0, \quad u(x,1) = -x.$

Solution 7b (Method of characteristics). We have

$$F \coloneqq xp^2 + yq$$

and the initial conditions are $x_0 = s$, $y_0 = 1$ and $u_0 = -s$. The additional initial conditions are found from the system

$$x_0 p_0^2 + y_0 q_0 = 0$$

and

$$u_0' = x_0' p_0 + y_0' q_0 \quad \Rightarrow \quad -1 = p_0.$$

Thus $p_0 = -1$ and $q_0 = -x_0 p_0^2 = -s$.

We have also the characteristic equations

$$\dot{x} = F_p = 2xp$$

$$\dot{y} = F_q = y$$

$$\dot{u} = pF_p + qF_q = 2xp^2 + yq = -yq$$

$$\dot{p} = -F_x - pF_u = -p^2$$

$$\dot{q} = -F_y - qF_u = -q.$$

Solve the last equation: $q = C_1 e^{-t}$, hence $q = q_0 e^{-t} = -se^{-t}$. The fourth equation gives

$$-\frac{dp}{p^2} = dt \quad \Rightarrow \quad t - \frac{1}{p} = C_2$$

We find $C_2 = -\frac{1}{p_0} = 1$, hence

$$p = \frac{1}{t - C_2} = \frac{1}{t-1}.$$

The second equation of the system gives immediately $y = y_0 e^t = e^t$ and the third then takes the form

$$\dot{u} = -yq = -e^t(-se^{-t}) = s \quad \Rightarrow \quad u = st + u_0 = st - s.$$

Finally we find x:

$$\dot{x} = 2xp = \frac{2x}{t-1} \quad \Rightarrow \quad \ln x = 2\ln(t-1) + C_3$$

We have $C_3 = \ln x_0 = \ln s$, whence

$$x = s(t-1)^2$$

Combining the found relations we obtain

$$u = \frac{x}{\ln y - 1}$$

Problem 9. Prove that the only solutions in all $\mathbb{R}^2$ to the equation

$$u^3 u_x' + u_y' = 0$$

are the constants.

Solution. Let $u(x, y)$ be any solution to this equation in the whole $\mathbb{R}^2$. The characteristics equations are

$$\dot{x} = u^3, \qquad \dot{y} = 1, \qquad \dot{u} = 0$$

That is $u = const = C$ and

$$x = C^3 t + x_0, \quad y = t + y_0$$

In particular, the solution is constant along the straight lines

$$x = C^3 t + x_0, \qquad y = t + y_0.$$

The explicit equation is

$$x = C^3(y - y_0) + x_0$$

If our solution $u(x, y)$ takes at least two different values C_1 and C_2 then the above lines must cross at some point in $\mathbb{R}^2$ (because the lines have different slope-coefficients). But in this case $u(x, y)$ takes both values C_1 and C_2 at the intersection point that contradicts to a single-valued character of $u(x, y)$. The contradiction shows that the solution must be a constant.

Problem 10. Prove that the initial-value problem

$$xu_x + yu_y = u^3, \qquad u(x, 0) = x,$$

has no solution. (*Hint*: differentiate the initial condition).

Solution. Arguing by contradiction, let us assume that $u(x, y)$ be such a solution. Then differentiating the initial condition we find

$$u_x(x, 0) = 1,$$

On the other hand, by virtue of the equation we have

$$xu_x(x, 0) + 0 \cdot u_y(x, 0) = x^3 \quad \Rightarrow \quad xu_x(x, 0) = x^3 \quad \Rightarrow \quad u_x(x, 0) = x^2.$$

A contradiction follows.