Partiella differentialekvationer: Exercise problems (2009-02-10)

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Problem 1. Find a first-order differential equation having the following solutions:

a) $u = xy + f(x - y)$
b) $u = x + y + f(xy)$
c) $u = ax^2 + xy + ay^2$

Here f is an arbitrary function and a is an arbitrary constant.

Problem 2. Find the general solution of the following homogeneous equations and draw the characteristic lines:

a) $u_x + yu_y = 0,$
b) $xu'_x + yu'_y + zu'_z = 0, \quad u = u(x, y, z)$

Problem 3. Find the general solution of the following equations:

a) $yu_x - xu_y = 1,$
b) $x^2u_x + y^2u_y = (x + y)u.$

Problem 4. Solve the following Cauchy problems by method of characteristics:

a) $yu_x + xu_y = 0, \quad u(0, y) = y^2$
b) $3u_x + 2u_y = 0, \quad u(x, 0) = \sin x$
c) $xu_x + yu_y = u + 1, \quad u(x, x^2) = x^2$

Problem 5. Solve the following Cauchy problems by method of Lagrange:

a) $u_x + uu_y = y, \quad u(0, y) = 1,$
b) $u_x + (x + y)u_y = 1, \quad u(x, -x) = 0.$

Problem 6. Solve the Cauchy problem

$$(1 + x^2)u'_x - 2xy\, u'_y = 0, \quad u(x, x + x^3) = h(x)$$

where $h(x)$ is some function.

Problem 7. Solve the initial-value problems for non-linear equations

a) $xu'_x + yu'_y + u'_x u'_y = u, \quad u(x, 1) = x^2$
b) $u_x = u_y^2, \quad u(0, y) = \frac{y^2}{2},$

by both characteristic method and the method of envelopes (try first affine solutions).

Problem 8. Find the solution to the Cauchy problem: $x{u'_x}^2 + yu'_y = 0, \quad u(x, 1) = -x.$

Problem 9. Prove that the only solutions in all $\mathbb{R}^2$ to the equation

$$u^3u'_x + u'_y = 0$$

are the constants. (*Hint*: find the characteristic lines and discuss when they form a non-intersecting family)

Problem 10. Prove that the initial-value problem

$$xu_x + yu_y = u^3, \qquad u(x, 0) = x,$$

has no solution. (*Hint*: differentiate the initial condition).