Solutions to some problems from Exercise problems (2009-02-20)

Problem 1. Find all solutions of $2uu_{xy} - u_x u_y = 1$ which satisfy the *ansatz* $u = f(x)g(y)$.

Solution. We have after substitution

$$f(x)f'(x)g(y)g'(y) = 1 \quad \Rightarrow \quad f(x)f'(x) = \frac{1}{g(x)g'(x)}$$

The last equality implies that both R.H.S. and L.H.S. are constants. Denote the common constant by c. Then we arrive to equations

$$f(x)f'(x) = c \quad \Rightarrow \quad f^2(x) = 2cx + a,$$

where a is an arbitrary constant. Similarly for g: $g^2(y) = \frac{2y}{c} + b$, where b is an arbitrary number. Plugging this into the ansatz we obtain

$$u = \sqrt{(2cx + a)\left(\frac{2y}{c} + b\right)} = 2\sqrt{(x + c_1)(y + c_2)}.$$

Problem 2a. Determine all characteristic curves to the following equations and transform the equation to normal form in the given set*:*

$$x^2 u_{xx} - u_{yy} = u, x \neq 0$$

Solution. The equation for characteristic curves in non-parametric form is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\pm 2x}{2x^2} = \pm\frac{1}{x}$$

Hence the equation is hyperbolic for $x \neq 0$ (the discriminant $b^2 - 4ac = 4x^2$ is positive there) and we find two solutions

$$y = \ln x + C_1, \qquad y = -\ln x + C_2\,.$$

We introduce new coordinates $\lambda = C_2 = y + \ln x$ and $\mu = C_1 = y - \ln x$. Then the xy-derivatives are

$$u'_x = u'_\lambda \lambda'_x + u'_\mu \mu'_x = \frac{u'_\lambda - u'_\mu}{x}$$

$$u''_{xx} = -\left(\frac{1}{x^2}\right)(u'_\lambda - u'_\mu) + \left(\frac{1}{x^2}\right)(u''_{\lambda\lambda} - 2u'_{\mu\lambda} + u''_{\mu\mu})$$

And similarly $u'_y = u'_\lambda + u'_\mu$ and $u''_{yy} = u''_{\lambda\lambda} + 2u'_{\mu\lambda} + u''_{\mu\mu}$. Then substitution these relations into our equation yields

$$\frac{x^2}{x^2}(-u'_\lambda + u'_\mu + u''_{\lambda\lambda} - 2u'_{\mu\lambda} + u''_{\mu\mu}) - (u''_{\lambda\lambda} + 2u'_{\mu\lambda} + u''_{\mu\mu}) = u,$$

Combining terms we find the normal form: $4u'_{\mu\lambda} + u'_\lambda - u'_\mu + u = 0.$

Problem 3b. *Reduce the equation to normal form and find its general solution*

$$x^2 u''_{xx} + 2xy\, u''_{xy} + y^2 u''_{yy} = y.$$

Solution. We follow the previous solution: $a = x^2, b = 2xy, c = y^2$ so that the discriminant is $b^2 - 4ac = 0$, hence our equation has parabolic type. The characteristic equation is

$$\frac{dy}{dx}=\frac{b}{2a}=\frac{2xy}{2x^2}=\frac{y}{x},$$

which implies $\frac{dy}{y}=\frac{dx}{x}$ and therefore $y=Cx$. We consider the new coordinates $\lambda=\frac{y}{x}$, $\mu=x$ (in the parabolic case one can choose the second variable **arbitrarily**). Hence

$$u'_x=u'_\lambda\lambda'_x+u'_\mu\mu'_x=-\frac{y}{x^2}u'_\lambda+u'_\mu$$

$$u''_{xx}=\frac{2y}{x^3}u'_\lambda-\frac{y}{x^2}\left(-\frac{y}{x^2}u''_{\lambda\lambda}+u''_{\mu\lambda}\right)+\left(-\frac{y}{x^2}u''_{\mu\lambda}+u''_{\mu\mu}\right)=\frac{2y}{x^3}u'_\lambda+\frac{y^2}{x^4}u''_{\lambda\lambda}-\frac{2y}{x^2}u''_{\mu\lambda}+u''_{\mu\mu}$$

$$u''_{xy}=-\frac{1}{x^2}u'_\lambda-\frac{y}{x^2}\left(\frac{1}{x}u''_{\lambda\lambda}\right)+\frac{1}{x}u''_{\lambda\mu}$$

$$u''_{yy}=\frac{1}{x^2}u''_{\lambda\lambda}$$

Substitution into $x^2u''_{xx}+2xy\,u''_{xy}+y^2u''_{yy}-y=0$ yields

$$x^2\left(\frac{2y}{x^3}u'_\lambda+\frac{y^2}{x^4}u''_{\lambda\lambda}-\frac{2y}{x^2}u''_{\mu\lambda}+u''_{\mu\mu}\right)+2xy\left(-\frac{1}{x^2}u'_\lambda-\frac{y}{x^2}\left(\frac{1}{x}u''_{\lambda\lambda}\right)+\frac{1}{x}u''_{\lambda\mu}\right)+\frac{y^2}{x^2}u''_{\lambda\lambda}-y=x^2u''_{\mu\mu}-y=0$$

In the new coordinates, the latter equation reads as

$$u''_{\mu\mu}=\frac{y}{x^2}=\frac{\lambda}{\mu}$$

The integration yields $u'_\mu=\lambda\ln\mu+C(\lambda)$ and then

$$u=\lambda(\mu\ln\mu-\mu)+C_1(\lambda)$$

Hence the general solution has the form

$$u=\frac{y}{x}(x\ln x-x)+f\left(\frac{y}{x}\right)=y\ln x-y+f\left(\frac{y}{x}\right),$$

where $f(t)$ is an arbitrary function of t.

Problem 4. Consider the following equation and answer the questions below:

$$2yu''_{xx}-2y^4u''_{xy}-3y^3\,u'_y=0.$$

(a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic.

Solution. (a) In our case $a=2y, b=0, c=-2y^4$. Hence $D=b^2-4ac=16y^5$ and the equation is *hyperbolic* when $y>0$ (and *elliptic* when $y<0$).

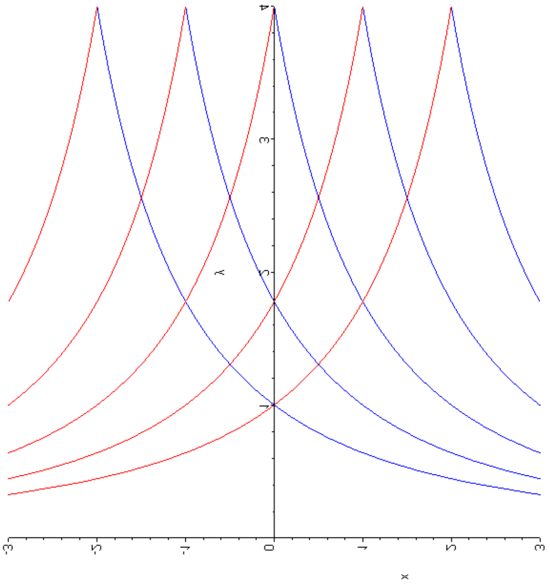
(b) To find the characteristic curves we write the corresponding ODE in the hyperbolic domain $y>0$:

$$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{0\pm\sqrt{16y^5}}{4y}=\pm y^{\frac{3}{2}}.$$

The latter yields

$$\frac{dy}{y^{3/2}}=\pm dx\ \Rightarrow\ -\frac{2}{\sqrt{y}}=\pm x+C$$

The characteristic curves are hyperbolas $-\frac{2}{\sqrt{y}} = x + C$ and $\frac{2}{\sqrt{y}} = -x + C$ (in the upper half-plane):



(c) In these new coordinates ⇒ $\lambda = x - \frac{2}{\sqrt{y}}$ and $\mu = -x - \frac{2}{\sqrt{y}}$ we find

$$u'_x = u'_\lambda - u'_\mu$$

$$u'_y = \frac{1}{y^{3/2}}(u'_\lambda + u'_\mu)$$

$$u''_{xx} = u''_{\lambda\lambda} - 2u''_{\lambda\mu} + u''_{\mu\mu}$$

$$u''_{yy} = -\frac{3}{2y^{5/2}}(u'_\lambda + u'_\mu) + \frac{1}{y^3}(u''_{\lambda\lambda} + 2u''_{\lambda\mu} + u''_{\mu\mu})$$

We find that

$$0 = 2yu''_{xx} - 2y^4 u''_{yy} - 3y^3\, u'_y$$

$$= 2y(u''_{\lambda\lambda} - 2u''_{\lambda\mu} + u''_{\mu\mu}) - 2y^4\left(\frac{u''_{\lambda\lambda} + 2u''_{\lambda\mu} + u''_{\mu\mu}}{y^3} - \frac{3(u'_\lambda + u'_\mu)}{2y^{\frac{5}{2}}}\right) - \frac{3(u'_\lambda + u'_\mu)}{y^{3/2}}$$

and, finally, $u''_{\lambda\mu} = 0$. The general solution to this equation is $u = f(\lambda) + g(\mu)$, the general solution to our equation is found as $u = F\left(x - \frac{2}{\sqrt{y}}\right) + G\left(x + \frac{2}{\sqrt{y}}\right)$, where F, G are arbitrary functions.

Problem 5c. By the reflection method find the solution of the initial-value problems in the wedge $x > 0, t > 0$: $u_{tt} - u_{xx} = x - t$, $u(x,0) = -x^2$, $u_t(x,0) = 3x^2$, $u(0,t) = \frac{t^2}{2}$ (*Hint*: find a solution to the nonhomogeneous equation and reduce the problem to the homogeneous equation.)

Solution. It is easy to find a partial solution of $u_{tt} - u_{xx} = x - t$. Since the R.H.S. is a sum of two functions, each depending on one variable, one can (the equation is linear!) find a solution by separation of variables. This gives

$$v(x,t) = -\frac{x^2}{2} - \frac{t^2}{2}.$$

Write $u = v + w$, where u is the solution to our Cauchy problem. Then w is a solution of

$$w_{tt} - w_{xx} = 0.$$

Moreover, we can translate the given Cauchy data into the new:

$$w(x,0) = u(x,0) - v(x,0) = -x^2 - \left(-\frac{x^2}{2}\right) = -\frac{x^2}{2} \equiv g(x),$$

$$w_t(x,0) = u_t(x,0) - v_t(x,0) = 3x^2 - 0 = 3x^2 \equiv h(x),$$

$$w(0,t) = u(0,t) - v(0,t) = \left(\frac{t^2}{2}\right) - \left(-\frac{t^2}{2}\right) = t^2 \equiv H(x).$$

We find first the solution for $x > t$ (by d' Alembert formula):

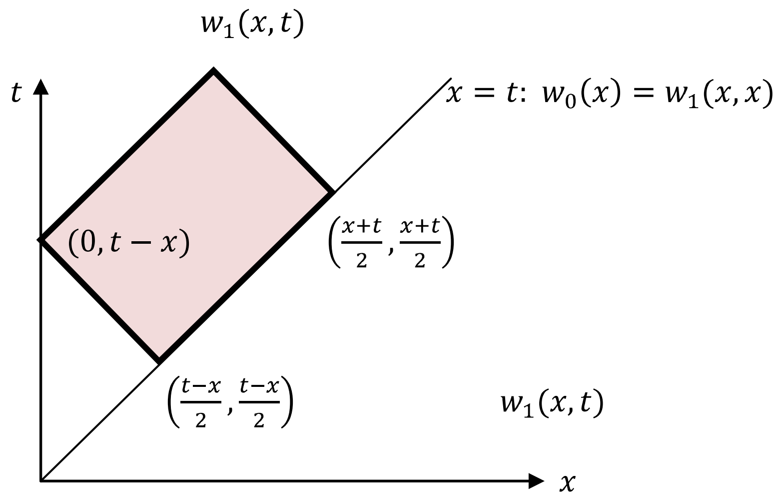
$$w_1(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds = t^3 + 3x^2t - \frac{x^2}{2} - \frac{t^2}{2}.$$

Applying the reflection method, we find then our solution for $x = t$:

$$w_0(x) = x^3 + 3x^3 - \frac{x^2}{2} - \frac{x^2}{2} = 4x^3 - x^2.$$

Then by the parallelogram rule, the solution in the upper wedge $x < t$ is found from (see the picture)

$$w_2(x,t) + w_0\left(\frac{t-x}{2}\right) = w_0\left(\frac{x+t}{2}\right) + H(t-x)$$



Hence $w_2(x,t) = x^2 + x^3 + t^2 - 3xt + 3xt^2$. Finally, u is found by the sum $v + w$:

$$u = t^3 + 3x^2t - x^2 - t^2, \qquad \text{for } x > t > 0,$$

$$u = \frac{x^2}{2} + x^3 + \frac{t^2}{2} - 3xt + 3xt^2, \qquad \text{for } t \geq x \geq 0.$$

Problem 7. By the Fourier method find solution of $u_{tt} - u_{xx} = 0$ with the initial data:

$$u(x,0) = 2\sin^2\left(x - \frac{\pi}{4}\right), \qquad u_t(x,0) = \sin 2x,$$
$$u(0,t) = 1, \qquad u(\pi,t) = 1$$

Solution. Set $v = u - 1$. Then v is the solution of the following problem:

$$v_{tt} - v_{xx} = 0, \qquad v(x,0) = 2\sin^2\left(x - \frac{\pi}{4}\right) - 1, \qquad v_t(x,0) = \sin 2x$$

$$v(0,t) = v(\pi,t) = 0.$$

Thus, we can apply the Fourier method of separation of variables. Simplify first the initial condition:

$$v(x,0) = 2\sin^2\left(x - \frac{\pi}{4}\right) - 1 = 2\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)^2 - 1 = -2\sin x\cos x = -\sin 2x.$$

We represent our solution by the formula

$$v(x,t) = a(t)\sin 2x.$$

(i.e. in our case the series contains only one term, due to our initial conditions). We find

$$a''(t) + 4a(t) = 0.$$

The general solution of the last equation is

$$a(t) = A\sin 2t + B\cos 2t\,.$$

We have $v(x,0) = -\sin 2x$, that is $a(0) = -1$. Similarly, $v_t(x,0) = \sin 2x$, hence $a'(0) = 1$. We find the constants A and B:

$$a(0) = B = -1, \qquad a'(0) = 2A = 1.$$

Finally, $a(t) = \frac{1}{2}\sin 2t - \cos 2t$, yields

$$v = \left(\frac{1}{2}\sin 2t - \cos 2t\right)\sin 2x.$$

We obtain $u = 1 + \left(\frac{1}{2}\sin 2t - \cos 2t\right)\sin 2x$.

Problem 9. Find the averages of the following functions on the 2-dimensional sphere $x^2 + y^2 + z^2 = 1$

a) $f = x^2$
b) $f = x^2y^2$,
c) $f = z^4$.

Solution. Denote by $\langle f\rangle$ the average of f over the sphere S. We shall exploit the following simple facts:

- the average is linear: $\langle f + g\rangle = \langle f\rangle + \langle g\rangle$,
- the average of a constant is equal the constant: $\langle c\rangle = c$,
- the sum $x^2 + y^2 + z^2$ is equal to 1 on the sphere.

a) We have $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle \;\Rightarrow\; 3\langle x^2\rangle = \langle x^2 + y^2 + z^2\rangle = \langle 1\rangle = 1$. Hence $\langle x^2\rangle = \frac{1}{3}$.

Solve b) and c). First notice that by symmetry,

$$\langle x^2z^2\rangle = \langle x^2y^2\rangle = \langle z^2y^2\rangle =: A, \qquad \langle x^4\rangle = \langle y^4\rangle = \langle z^4\rangle =: B.$$

Next, we multiply x^2 by $1 \equiv x^2 + y^2 + z^2$, then applying (a) and take the average:

$$\frac{1}{3} = \langle x^2\rangle = \langle x^2(x^2 + y^2 + z^2)\rangle = \text{by linearilty} =$$

$$= \langle x^4\rangle + \langle x^2y^2\rangle + \langle x^2z^2\rangle = B + 2A.$$

Hence $2A + B = 1/3$. In order to find another relation we introduce the new variables

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, \qquad y' = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y, \qquad z' = z.$$

(This change of variables corresponds to the rotation in the xy-plane by angle 45^o). Then

$$2x'y' = x^2 - y^2 \quad \Rightarrow \quad 4x'^2y'^2 = x^4 - 2x^2y^2 + y^4,$$

and since the average is invariant with respect to rotation, we have

$$4A = 4\langle x^2y^2\rangle \equiv 4\langle x'^2y'^2\rangle = \langle x^4 - 2x^2y^2 + y^4\rangle = 2B - 2A,$$

hence $6A = 2B$. Combining the found relations we find

$$\langle x^2y^2\rangle = A = \frac{1}{15}, \qquad \langle x^4\rangle = B = \frac{1}{5}.$$

Problem 10b. Find the solution of the 3-dim wave equation $u''_{tt} = u''_{xx} + u''_{yy} + u''_{zz}$ with Cauchy data

$$u(x,y,z,0) = x^2y^2, \qquad u_t(x,y,z,0) = z^2.$$

Solution. We shall apply the solution of Problem 9. We have for the averages (here x, y, z are parameters and ξ_k are the Euclidean coordinates):

$$M_g = \langle g(x + t\xi_1, y + t\xi_2, z + t\xi_2)\rangle = \langle (x + t\xi_1)^2(y + t\xi_2)^2\rangle = \text{by linearilty} =$$

$$= x^2y^2\langle 1\rangle + t^2x^2\langle \xi_2^2\rangle + t^2y^2\langle \xi_1^2\rangle + t^4\langle \xi_1^2\xi_2^2\rangle = \text{by Problem 9}$$

$$= x^2\, y^2 + \frac{t^2x^2}{3} + \frac{t^2y^2}{3} + \frac{t^4}{15}$$

Similarly,

$$M_h = \langle h(x + t\xi_1, y + t\xi_2, z + t\xi_2)\rangle = \langle (\,z + t\xi_2)^2\rangle = z^2 + 2zt\langle \xi_3\rangle + t^2\langle \xi_3^2\rangle = z^2 + \frac{t^2}{3}.$$

Applying Kirchhoff's formula $u(x,t) = \frac{\partial}{\partial t}\left(t\, M_g\right) + tM_h$

yields

$$u = \left(t\left(x^2\, y^2 + \frac{t^2x^2}{3} + \frac{t^2y^2}{3} + \frac{t^4}{15}\right)\right)'_t + t\left(z^2 + \frac{t^2}{3}\right)$$

and finally,

$$u = x^2y^2 + tz^2 + t^2(x^2 + y^2) + \frac{t^3}{3} + \frac{t^4}{3}.$$

Problem 11. Find the solution of the 2-dim wave equation $u''_{tt} = u''_{xx} + u''_{yy}$ with Cauchy data

$$u(x,y,0) = xy, \quad u_t(x,y,0) = x^2.$$

Solution. We follow the method of Problems 9-10. Here we interpret the solution $u(x,y,t)$as the solution to the three-dimensional wave equation which is independent of z. We repeat the steps given in the solution of Problem 10.

$$M_g = \langle (x + \xi_1)(\,y + t\xi_2)\rangle = xy + 0 + 0 + 0 = xy,$$

$$M_h = \langle (x + t\xi_1)^2\rangle = x^2 + t^2\langle \xi_1^2\rangle = x^2 + \frac{t^2}{3}.$$

Applying Kirchhoff's formula we find $u = xy + tx^2 + \frac{t^3}{3}$.