**Partiella differentialekvationer: Exercise problems (2009-02-20)**

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1. Find all solutions of $2uu_{xy} - u_x u_y = 1$ which satisfy the *ansatz* $u = f(x)g(y)$.
2. Determine all characteristic curves to the following equations and transform the equation to normal form:
   a. $x^2 u''_{xx} - u''_{yy} = u, \quad x \neq 0$
   b. $x^2 u''_{xx} - 2\sin x\, u''_{xy} - \cos^2 x\ u''_{yy} - \cos x\ u'_y = 0$
   c. $y^2 u''_{xx} + x^2 u''_{yy} = 0, \qquad x > 0,\ y > 0$
3. Reduce the equations to the normal form and find its general solution
   a) $yu_{xx} + 3yu_{xy} + 3u_x = 0,$
   b) $x^2 u''_{xx} + 2xy\, u''_{xy} + y^2 u''_{yy} = y,$
   c) $u_{xx} - 4u_{xy} + 4u_{yy} = e^y.$
   d) $x^2 u''_{xx} - 2\sin x\, u''_{xy} - \cos^2 x\ u''_{yy} - \cos x\ u'_y = 0$
4. Consider the following equation and answer the questions below:
   $$2yu''_{xx} - 2y^4 u''_{xy} - 3y^3\, u'_y = 0.$$
   (a) Where is the equation hyperbolic? (b) Determine the characteristic curves. (c) Transform the equation to canonical form where this is possible. (d) Determine its general solution in the domain where it is hyperbolic.

5. By the reflection method find the solution of the initial-value problems in the wedge $x > 0, t > 0$:
   a) $u_{tt} - u_{xx} = 0,\ u(x,0) = x,\ \ u_t(x,0) = 3x^2,\ \ u(0,t) = 1$
   b) $u_{tt} - u_{xx} = 0,\ u(x,0) = e^{-x^2},\ \ u_t(x,0) = 0,\ \ u(0,t) = 1$
   c) $u_{tt} - u_{xx} = x - t,\ \ u(x,0) = -x^2,\ \ u_t(x,0) = 3x^2,\ \ u(0,t) = \frac{t^2}{2}$ (*Hint*: find a solution to the nonhomogeneous equation and reduce the problem to the homogeneous equation.)

6. Solve the initial value problem $u(x,0) = x, u_t(x,0) = 3x^2$ for
   $$u_{tt} - u_{xx} = x^2 - t, \qquad x \in \mathbb{R},\ \ t > 0.$$
7. By the Fourier method find solution of $u_{tt} - u_{xx} = 0$ with the initial data:
   $$u(x,0) = 2\sin^2\left(x - \frac{\pi}{4}\right), \qquad u_t(x,0) = \sin 2x,$$
   $$u(0,t) = 1, \qquad u(\pi,t) = 1$$
   *Hint*: consider $v = u - 1$.

8. Solve the differential equation $u_{tt} = u_{xx} + u_x + u_t$ with the initial conditions $u(x,0) = x$, $u'_t(x,0) = 0$.
   *Hint*: Find $a$ and $b$ such that $v(x,y) = e^{at+bx}u(x,y)$ solves the ordinary wave equation.

9. Find the averages of the following functions on the two-dimensional sphere $x^2 + y^2 + z^2 = 1$
   a) $f = x^2$
   b) $f = x^2y^2,$
   c) $f = z^4.$
10. Find the solution of the 3-dim wave equation $u''_{tt} = u''_{xx} + u''_{yy} + u''_{zz}$ with Cauchy data
    a) $u(x,y,z,0) = x^4,\ \ u_t(x,y,z,0) = 0$
    b) $u(x,y,z,0) = x^2y^2,\ \ u_t(x,y,z,0) = z^2$
11. Find the solution of the 2-dim wave equation $u''_{tt} = u''_{xx} + u''_{yy}$ with Cauchy data
    c) $u(x,y,0) = xy,\ \ u_t(x,y,0) = x^2.$