Partiella differentialekvationer: Exercise problems (2009-03-10)

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1. Solve the boundary value problem for the Laplace equation in a square

$$u''_{xx} + u''_{yy} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1,$$
$$u(0, y) = u(x, 0) = 0, \qquad u(1, y) = 2y, \quad u(x, 1) = 3 \sin \pi x + 2x.$$

2. Show that $\frac{1}{|x|}$ is locally integrable in $\mathbb{R}^3$ ($|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$) and find

$$\lim_{\epsilon \to +0} \int_{|x|>\epsilon} \frac{\Delta\phi(x)}{|x|}\, dx_1 dx_2 dx_3$$

for any twice-differentiable function ϕ with compact support.

3. Find the radial symmetric *continuous* solution of the Poisson equation

$$\Delta u = 1, \qquad x \in \mathbb{R}^3,$$

satisfying $u(1,2,1) = 1$.

4. Find all solutions of the heat equation

$$u_{xx} - u_t = 0$$

which have the form $u = t^{-\frac{1}{2}} f(t^\alpha x^2)$, where f is a function and α is a real number to be found.
(Hint: reduce to a second order ODE with respect to $\xi = t^\alpha x^2$ and find the possible values of α.)

5. Solve the pure initial value problem for the heat equation

$$u_{xx} + u_{yy} = u_t, \qquad u(x, y, 0) = e^{-x^2-y^2}.$$

Find the maximal temperature at time $t = 2$.