Partiella differentialekvationer: Solutions to 2009-03-10

Problem 1. Solve the boundary value problem for the Laplace equation in a square

$$u''_{xx} + u''_{yy} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1,$$
$$u(0,y) = u(x,0) = 0, \qquad u(1,y) = 2y, \quad u(x,1) = 3\sin \pi x + 2x.$$

Solution. First we reduce the problem to a homogeneous one. Notice that $2xy$ is a harmonic function and then $u_0 = u - 2xy$ is also harmonic. The boundary conditions for u_0 are found as

$$u_0(0,y) = u(0,y) = 0,$$

$$u_0(x,0) = u(x,0) = 0,$$

$$u_0(1,y) = u(1,y) - 2y = 0,$$

$$u_0(x,1) = u(x,1) - 2x = 3\sin \pi x.$$

Now we can apply the method of separation of variables to u_0: we try find the solution in the form

$$u_0(x,y) = \sum_{n=1}^{\infty} A_n v_n(x) w_n(y),$$

where v_n and $w_n$the "elementary solutions", that is

$$\Delta\big(v(x)w(y)\big) = 0.$$

The latter implies $v''(y)w(y) + v(x)w''(y) = 0$, and consequently $\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = C$, where C is some constant. Applying our boundary conditions $v(0) = v(1) = 0$ we obtain that v must be a trigonometric function, moreover a sinus function:

$$v_n(x) = \sin \pi n x\,, \qquad n = 1,2,3, \ldots$$

This function corresponds to $C_n = -(\pi n)^2$. We have then for the second component

$$w'' - (\pi n)^2 w = 0,$$

that is the solution is found by means of the hyperbolic functions:

$$w_n(y) = A_n \cosh \pi n y + B_n \sinh \pi n y.$$

Then the boundary condition $w(0) = 0$ implies $A_n = 0$. In summary, we have

$$u_0(x,y) = \sum_{n=1}^{\infty} A_n \sin \pi n x \sinh \pi n y.$$

It remains only to find the coefficients. Notice that

$$u_0(x,1) = 3\sin \pi x = \sum_{n=1}^{\infty} (A_n \sinh \pi n) \sin \pi n x,$$

hence $A_n \sinh \pi n = 3$ for $n = 1$ and $A_n = 0$ otherwise. This gives

$$u_0(x,y) = \frac{3}{\sinh \pi} \sin \pi x \sinh \pi y,$$

and finally

$$u = u_0 + 2xy = 2xy + 3 \sin \pi x \frac{\sinh \pi y}{\sinh \pi}.$$

Problem 2. Show that $\frac{1}{|x|}$ is locally integrable in $\mathbb{R}^3$ ($|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$) and find

$$\lim_{\epsilon \to +0} \int_{|x|>\epsilon} \frac{\Delta\phi(x)}{|x|}\, dx_1 dx_2 dx_3$$

for any twice-differentiable function ϕ with compact support.

Solution. Because $f \equiv \frac{1}{|x|}$ is continuous everywhere in the punctured space $\mathbb{R}^3 \setminus \{0\}$ then it is locally integrable there. It remains only to show that f is integrable in any ball

$$B_R = \{x: |x| < R\}.$$

This is equivalent to the existence of absolute convergence of the integral

$$I_\epsilon = \int_{R>|x|>\epsilon} \frac{1}{|x|}\, dx_1 dx_2 dx_3.$$

We calculate this by the spherical Fubini theorem:

$$I_\epsilon = \int_\epsilon^R dr \int_{S_r} \frac{dS}{|x|} = \int_\epsilon^R dr \int_{S_r} \frac{dS}{r} = \int_\epsilon^R \frac{dr}{r} \mathrm{Area}(S_r) = 4\pi \int_\epsilon^R r dr = 2\pi(R^2 - \epsilon^2).$$

(S_r is the sphere of radius r and with center at the origin). We see that I_ϵ converges as $\epsilon \to 0$, hence we have proved that $f = \frac{1}{|x|}$ is locally integrable.

In order to find the limit we apply the 2nd Green identity

$$\int_\Omega (v\, \Delta u - u\, \Delta v)\, dx = \int_{\partial\Omega} v\, \frac{\partial u}{\partial \nu} - u\, \frac{\partial v}{\partial \nu}\, dS$$

which is valid for any functions u, v continuous in $\overline{\Omega}$ and twice differentiable in the interior Ω. We take $u = \phi$ and $v = \frac{1}{|x|}$. We notice also that $f = \frac{1}{|x|}$ is harmonic. Indeed,

$$\nabla f = \nabla \frac{1}{r} = -\frac{1}{r^2} \nabla r = -\frac{1}{r^2} \cdot \frac{x}{r},$$

$$\Delta \frac{1}{r} = -\mathrm{div}\left(\frac{1}{r^2} \cdot \frac{x}{r}\right) = -\frac{1}{r^3}\, \mathrm{div}\, x + \langle x, \frac{3x}{r^5} \rangle = -\frac{3}{r^3} + \frac{3}{r^3} = 0.$$

Since ϕ has compact support, it vanishes outside a large ball, say in B_R. Then the Green identity for $\Omega = B_R \setminus B_\epsilon$ reads as

$$\int_{\Omega} \Delta\phi(x)\,\frac{dx}{|x|} = \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}\cdot\frac{dS}{r} - \int_{\partial\Omega} \phi(x)\,\partial_\nu\left(\frac{1}{r}\right)\,dS = -\frac{1}{\epsilon}\int_{|x|=\epsilon} \frac{\partial\phi}{\partial\nu}\,dS - \frac{1}{\epsilon^2}\int_{|x|=\epsilon} \phi\,dS$$

Here we used that $\phi(R) = \partial_\nu\phi(R) = 0$ and $\partial_\nu\left(\frac{1}{r}\right) = -\frac{1}{r^2}$. We have for the integrals: the first integral converges to zero because $\frac{\partial\phi}{\partial\nu}$ is bounded in the unit ball (say by constant M) and the surface integral $\int_{|x|=\epsilon}\frac{\partial\phi}{\partial\nu}\,dS$ then can be estimated as

$$\left|\frac{1}{\epsilon}\int_{|x|=\epsilon} \frac{\partial\phi}{\partial\nu}\,dS\right| \le M\cdot\frac{4\pi\epsilon^2}{\epsilon} \to 0.$$

For the second integral: for any small δ we chose ϵ such that $|\phi(x)-\phi(0)| < \delta$ if $|x| < \epsilon$ (this is possible since ϕ is continuous at the origin). Then

$$\left|\frac{1}{\epsilon^2}\int_{|x|=\epsilon} \phi(x)\,dS - 4\pi\phi(0)\right| = \left|\frac{1}{\epsilon^2}\int_{|x|=\epsilon} \big(\phi(x)-\phi(0)\big)\,dS\right| \le 4\pi\delta.$$

This shows that $\frac{1}{\epsilon^2}\int_{|x|=\epsilon}\phi(x)\,dS \to 4\pi\phi(0)$ as $\epsilon \to 0$.

Thus we have

$$\lim_{\epsilon\to+0}\int_{|x|>\epsilon} \frac{\Delta\phi(x)}{|x|}\,dx_1dx_2dx_3 = -4\pi\phi(0).$$

Problem 3. Find the radial symmetric *continuous* solution of the Poisson equation

$$\Delta u = 1, \qquad x \in \mathbb{R}^3,$$

satisfying $u(1,2,1) = 1$.

Solution. Denote $u(x) = f(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then

$$\nabla f(r) = f'(r)\nabla r = f'\cdot\frac{x}{r},$$

$$\Delta f = \operatorname{div}\left(\frac{f'(r)}{r}x\right) = \frac{3f'(r)}{r} + \left(\frac{f'(r)}{r}\right)' r = f'' + \frac{2}{r}f'.$$

We arrive at the equation

$$f'' + \frac{2}{r}f' = 1,$$

and the Cauchy condition must be determined now by $u(1,2,1) = 1$. We have

$$r(1,2,1) = \sqrt{1+4+1} = \sqrt{6},$$

hence

$$u(1,2,1) = f\big(\sqrt{6}\big) = 1.$$

Solve the above differential equation by substitution $y = f'$:

$$y' + \frac{2}{r}y = 1, \qquad y = \frac{r}{3} + \frac{C_1}{r^2} \equiv f', \qquad f = \frac{r^2}{6} - \frac{C_1}{r} + C_2.$$

Since the solution must be continuous, it follows that $C_1 = 0$. Moreover, by the Cauchy data,

$$f(\sqrt{6}) = \frac{6}{6} + C_2 = 1, \qquad C_2 = 0.$$

We find finally

$$u = \frac{r^2}{6} = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2).$$

Problem 4. Find all solutions of the heat equation

$$u_{xx} - u_t = 0$$

which have the form $u = t^{-\frac{1}{2}} f(t^\alpha x^2)$, where f is a function and α is a real number to be found. *(Hint:* reduce to a second order ODE with respect to $\xi = t^\alpha x^2$ and find the possible values of α.)

Solution. Denote $\xi = t^\alpha x^2$ and find the derivatives:

$$u_t = -\frac{1}{2} t^{-\frac{3}{2}} f(\xi) + \alpha t^{\alpha - \frac{3}{2}} x^2 f'(\xi) = t^{-\frac{3}{2}}\left(-\frac{1}{2} f(\xi) + \alpha \xi f'(\xi)\right),$$

$$u_x = 2x t^{\alpha - \frac{1}{2}} f'(\xi),$$

$$u_{xx} = 4x^2 t^{2\alpha - \frac{1}{2}} f''(\xi) + 2t^{\alpha - \frac{1}{2}} f'(\xi).$$

Substitution into $u_{xx} - u_t = 0$ then yields

$$4\xi t^{\alpha+1} f''(\xi) + 2t^{\alpha+1} f'(\xi) = -\frac{1}{2} f(\xi) + \alpha \xi f'(\xi),$$

hence

$$t^{\alpha+1} = \frac{-\frac{1}{2} f(\xi) + \alpha \xi f'(\xi)}{4\xi f''(\xi) + 2f'(\xi)}$$

The right hand side of the latter equation depends on $\xi = x^2 t^\alpha$ which is independent variable with respect to t , hence $\alpha + 1 = 0$. We have $\alpha = -1$, therefore in summary,

$$8\xi f''(\xi) + 4f'(\xi) = -\big(f(\xi) + 2\xi f'(\xi)\big)$$

Dividing both sides by $\xi^{\frac{1}{2}}$ we combine the terms by the Lebnitz product formula

$$8\xi^{1/2} f''(\xi) + 4\xi^{-1/2} f'(\xi) = 8\left(\xi^{\frac{1}{2}} f'(\xi)\right)',$$

$$\xi^{-1/2} f(\xi) + 2\xi^{1/2} f'(\xi) = 2\left(\xi^{\frac{1}{2}} f(\xi)\right)'.$$

We have

$$8\left(\xi^{\frac{1}{2}} f'(\xi)\right)' = -2\left(\xi^{\frac{1}{2}} f(\xi)\right)',$$

and after integration

$$4f'(\xi)+f(\xi)=C_1\xi^{-\frac{1}{2}}.$$

The solution of the homogeneous equation is

$$f_0=C_2e^{-\frac{\xi}{4}},$$

and the solution of the nonhomogeneous is found by variation of constants. We find finally the required solution in terms of the Gauss error integral

$$f=C_2e^{-\frac{\xi}{4}}+C_1e^{-\frac{\xi}{4}}\int^{\xi}e^{\frac{\xi}{4}}\xi^{-\frac{1}{2}}\,d\xi=C_2e^{-\frac{\xi}{4}}+C_3e^{-\frac{\xi}{4}}\int^{\sqrt{\xi}}e^{\frac{s^2}{4}}\,ds.$$

It remains only to substitute $\xi=\frac{x^2}{t}$.

Problem 5. Solve the pure initial value problem for the heat equation

$$u_{xx}+u_{yy}=u_t,\qquad u(x,y,0)=e^{-x^2-y^2}.$$

Find the maximal temperature at time $t=2$.

Solution. We have the general representation

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-\xi|^2}{4t}}g(\xi)d\xi.$$

In our case, $n=2$, $x=(x_1,x_2)$, $\xi=(\xi_1,\xi_2)$ and $g(x_1,x_2)=e^{-x_1^2-x_2^2}$. Thus

$$u(x,t)=\frac{1}{4\pi t}\int_{\mathbb{R}^2}e^{-\frac{|x-\xi|^2}{4t}}e^{-\xi_1^2-\xi_2^2}\,d\xi_1d\xi_2$$

We have

$$|x-\xi|^2=|x|^2-2\langle x,\xi\rangle+|\xi|^2,$$

and for the integrand:

$$e^{-\frac{|x-\xi|^2}{4t}}e^{-\xi_1^2-\xi_2^2}=\exp\left(-\frac{|x|^2-2\langle x,\xi\rangle+|\xi|^2}{4t}-|\xi|^2\right).$$

Simplify, by completing the square:

$$\frac{|x|^2-2\langle x,\xi\rangle+|\xi|^2}{4t}+|\xi|^2=\frac{|x|^2}{4t}+|\xi|^2\left(1+\frac{1}{4t}\right)-\frac{2\langle x,\xi\rangle}{4t}$$

$$=\frac{|x|^2}{4t}+\left|\xi\sqrt{1+\frac{1}{4t}}-\frac{x}{4t\sqrt{1+\frac{1}{4t}}}\right|^2-\left|\frac{x}{4t\sqrt{1+\frac{1}{4t}}}\right|^2=$$

$$= \frac{|x|^2}{4t+1} + |\xi a - b|^2,$$

where $a = \sqrt{1 + \frac{1}{4t}}$ and $b = \frac{x}{4t\sqrt{1+\frac{1}{4t}}}$. This gives

$$u(x,t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t+1}} \cdot \int_{\mathbb{R}^2} e^{-|\xi a - b|^2} d\xi_1 d\xi_2 =$$

(change the variables $\eta = \xi a - b$, the determinant of the Jacobi matrix is a^2)

$$= \frac{1}{4\pi t a^2} e^{-\frac{|x|^2}{4t+1}} \cdot \int_{\mathbb{R}^2} e^{-|\eta|^2} d\eta_1 d\eta_2 = \frac{1}{4\pi t a^2} e^{-\frac{|x|^2}{4t+1}} \cdot \left(\sqrt{\pi}\right)^2$$

$$= \frac{1}{4t\left(1 + \frac{1}{4t}\right)} e^{-\frac{|x|^2}{4t+1}} = \frac{1}{4t+1} e^{-\frac{|x|^2}{4t+1}}.$$

This is the required solution.

Now the maximal temperature for $t = 2$ is found by

$$u(x,2) = \max_{\forall x} \left(\frac{1}{9} e^{-\frac{|x|^2}{9}} \right) = \frac{1}{9}.$$