TATA57: Transform Theory VT 2020

Extended Lecture notes

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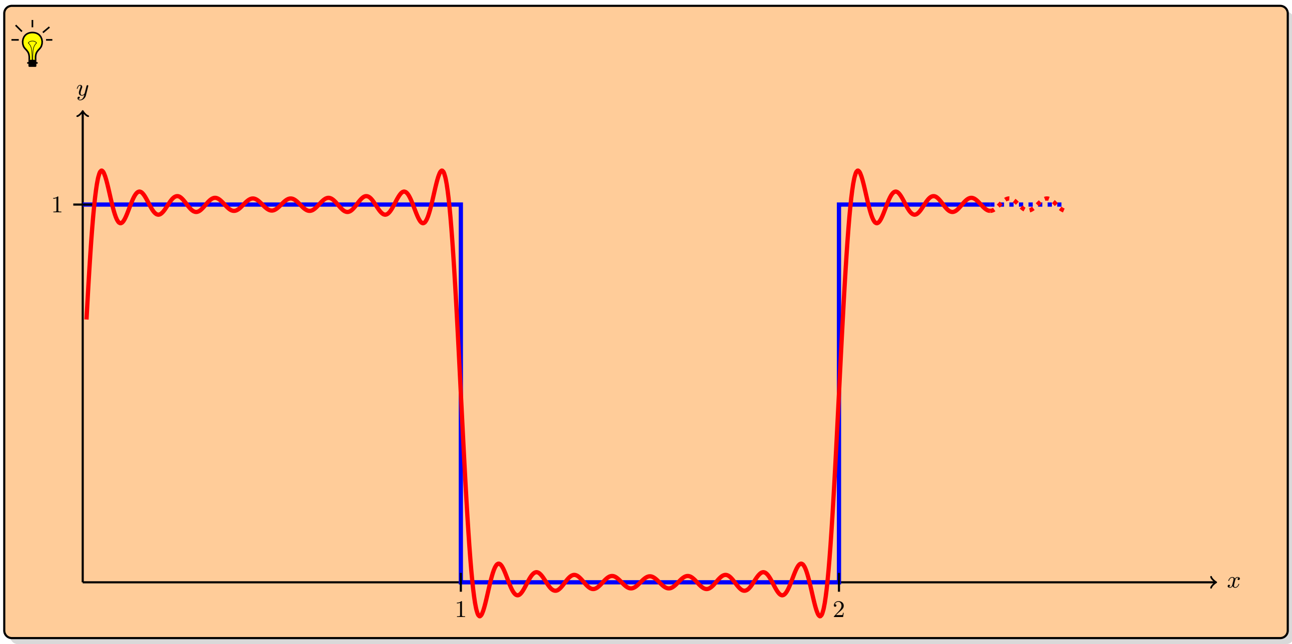

August 14, 2020

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Chapter 1

Periodic Functions, Series and Fourier Series

"It's Showtime!"
—Ben Richards

1.1 Preliminaries

The prerequisites for this course is basically single variable analysis, multivariate analysis and linear algebra. Some complex analysis is helpful but I'll make the course self-contained with respect to that.

1.1.1 Complex-valued Functions

We will immediately start working with complex valued functions of a real variable (at this point, we'll consider complex valued functions of a complex variable later on). If you've taken a course in complex analysis, everything will be familiar. If not, we do not need too much complex analysis (although complex numbers will be everywhere). Let's make a couple of general definitions for the things that we will need.

Definition. We write that $\lim_{z \to z_0} f(z) = A$ for some $A \in \mathbf{C}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Rightarrow \quad |f(z) - f(z_0)| < \epsilon.$$

We call f continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

So the definition is almost identical with the real case, it's just that $| \cdot |$ is now the complex absolute value (meaning that $|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$). Similarly to the real case, continuity can equivalently be phrased in terms of sequences (Heine's definition): for any sequence $z_n \to z_0$ we have $f(z_n) \to f(z_0)$. This description is sometimes easier to deal with than Cauchy's δ-ϵ-definition.

At this point, we will mainly consider functions $u\colon \mathbf{R} \to \mathbf{C}$. For functions of this type, we can always write $u(x) = \alpha(x) + i\beta(x)$, where $\alpha, \beta\colon \mathbf{R} \to \mathbf{R}$ are real-valued functions (the real and imaginary part of $u(x)$). Operations like differentiation and integration works like expected. We treat the real and imaginary part separately and then sum the results, i.e.,

$$u'(x) = \alpha'(x) + i\beta'(x) \quad \text{and} \quad \int_a^b u(x)\,dx = \int_a^b \alpha(x)\,dx + i\int_a^b \beta(x)\,dx.$$

This simplifies matters. In the case when we need to consider functions of a complex variable, things get a bit trickier, but that can wait until the second half of the course. This decomposition into real- and imaginary parts of the function $u(x)$ is sufficient for what we need right now.

1.2 Periodic Functions

A function $u\colon \mathbf{R} \to \mathbf{C}$ is called **periodic** if there is some constant $T > 0$ such that

$$u(x+T) = u(x) \text{ for every } x \in \mathbf{R}.$$

Note that if u is T-periodic, then u is also $2T$-periodic since

$$u(x+2T) = u(x+T+T) = u(x+T) = u(x) \text{ for every } x \in \mathbf{R}.$$

And similarly, u is nT periodic for $n = 1, 2, 3, \ldots$. We usually refer to the smallest possible period T when referring to a function's period. A constant function does not have a smallest period (but is obviously periodic).

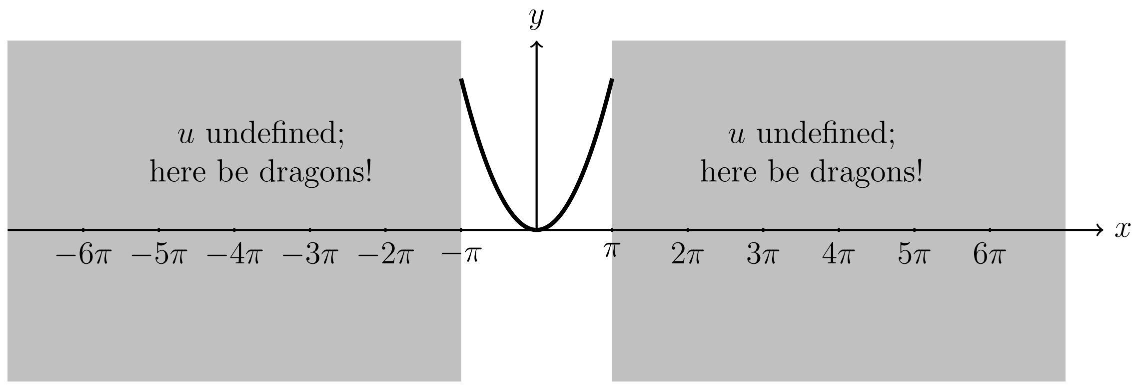
Example

(i) The functions $\sin t$ and $\cos t$ are 2π-periodic functions.

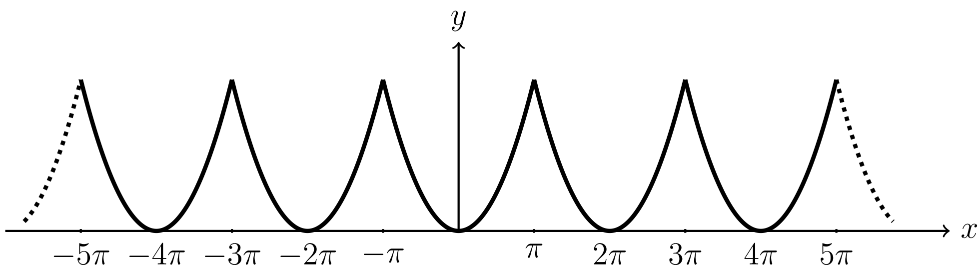
(ii) The functions e^{int} are $\dfrac{2\pi}{n}$-periodic functions.

These functions are usually known as *harmonic oscillations*.

In this course, we will mainly be considering 2π-periodic functions. How would we handle a function that is not periodic? Consider a function $u\colon [-\pi, \pi] \to \mathbf{C}$. This means that u is undefined outside the interval $[-\pi, \pi]$. For example, the graph (for a real example) could look something like below.



From a function $u\colon [a, b] \to \mathbf{C}$ defined on an interval $[a, b]$ (say $[-\pi, \pi]$), we can consider the periodic extension of u that is defined for all $x \in \mathbf{R}$ such that $u(x+T) = u(x)$ for every x, where $T = b - a$. For the function above, the periodic extension would look like the graph below.



Integrating periodic functions

If u is an integrable periodic function with period T, note that $\int_0^T u(x)\,dx = \int_a^{a+T} u(x)\,dx$ for any $a \in \mathbf{R}$. Therefore we can choose any integration domain of length T and to make the notation more compact, we often write $\int_T u(x)\,dx$ to indicate that we integrate over one period of the function.

1.3 Function Spaces

Let's start with defining two rather general spaces.

$L^1(a, b)$

Definition. We define the space $L^1(a, b)$ to consist of those functions $u\colon]a, b[\to \mathbf{C}$ for which

$$\int_a^b |u(x)|\,dx < \infty.$$

In other words, we collect those functions that are absolutely integrable on $[a, b]$.

$L^2(a, b)$

Definition. We define the space $L^2(a, b)$ to consist of those functions $u\colon]a, b[\to \mathbf{C}$ for which

$$\int_a^b |u(x)|^2\,dx < \infty.$$

These definitions might look fairly innocuous, but there's some stuff buried here. First and foremost, we really should be using a different type of integral than the Riemann integral we're used to (the Lebesgue counterpart). But in the case where the function is Riemann integrable, these two integrals coincide so we can live with this problem in this course. There's more issues hiding around the corner, and we'll get to some of these next lecture. The way we will handle this in this course is to restrict our attention to a subset of $L^2(a, b)$ where these problems are nonexistent.

Piecewise continuous function

Definition. We call a function u on an interval $[a,b]$ piecewise continuous if there are a finite number of points such that u is continuous everywhere on $[a,b]$ except for at these points. Moreover, if $c \in]a,b[$ is one of these points, the limits

$$\lim_{x\to c^-} u(x) \quad \text{and} \quad \lim_{x\to c^+} u(x)$$

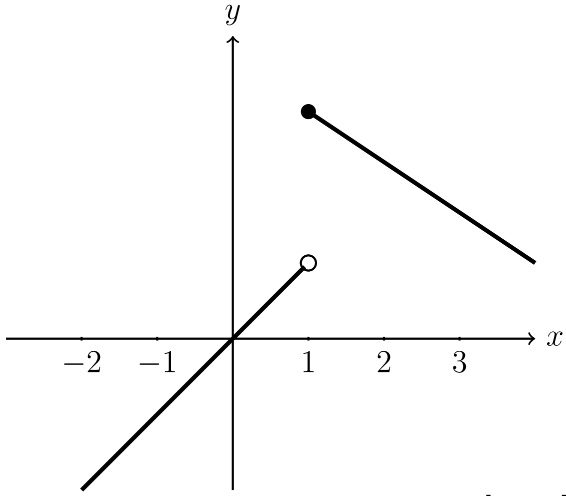
exist. We denote the space of all piecewise continuous functions on an interval $[a,b]$ by $E[a,b]$, or just E if the interval is clear from the context.

We will denote the left- and righthand limits at a point c by

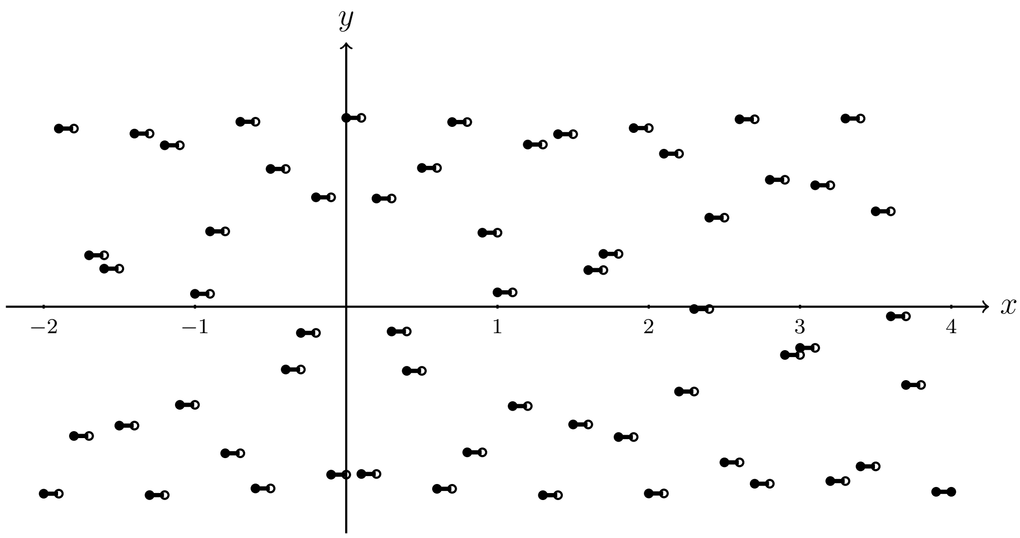
$$u(c^-) = \lim_{x\to c^-} u(x) \quad \text{and} \quad u(c^+) = \lim_{x\to c^+} u(x),$$

respectively.
As an example, we could consider the function

$$f(x) = \begin{cases} x, & -2 \le x < 1, \\ 4-x, & 1 \le x \le 3. \end{cases}$$



We might consider something more dramatic as well. The function below is in $E[-2,4]$ (it is in fact even piecewise constant).



So you probably get the point. We can cover quite a large amount of function types by only considering piecewise continuous functions. However, this thinking might be a bit disingenuous. It should be noted that this class of functions is still extremely small compared to, say, $L^2(2,4)$.

1.3.1 Left- and Righthand Derivatives

For $u \in E$, we define the left- and righthand derivatives at a point $x \in]a, b[$ by

$$D^- u(x) = \lim_{h \to 0^-} \frac{u(x+h) - u(x^-)}{h} \quad \text{and} \quad D^+ u(x) = \lim_{h \to 0^+} \frac{u(x+h) - u(x^+)}{h}$$

if the limit exist. For the endpoints, we only define $D^+ u(a)$ and $D^- u(b)$.

The space $E'[a, b]$

Definition. The linear space $E'[a, b]$ consists of those $u \in E[a, b]$ such that $D^- u(x)$ exists for $a < x \leq b$ and that $D^+ u(x)$ exists for $a \leq x < b$.

Note the following.

Properties

(i) If u is continuous, then $u \in E$.

(ii) If u is differentiable, then $u \in E'$.

(iii) On a compact interval, $E' \subset E \subset L^2 \subset L^1$ (that $L^2 \subset L^1$ follows from Cauchy-Schwarz).

1.4 Series

As we remember from TATA42, we define a numerical series S of a sequence $a_0, a_1, a_2, \ldots$ by

$$S = \sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k$$

whenever this limit exists (this is the *definition* of a convergent series). We have also studied certain types of *functional series*:

$$S(x) = \sum_{k=0}^{\infty} u_k(x) = \lim_{n \to \infty} \sum_{k=0}^{n} u_k(x)$$

for those x where the limit exists. In particular, we've seen power series where $u_k(x) = c_k x^k$ and c_k are real (or complex) constants. The sums

$$S_n(x) = \sum_{k=0}^{n} u_k(x), \quad n \in \mathbf{N},$$

are called the **partial sums** of the series S. Whenever $S_n(x)$ has a limit as $n \to \infty$, this is the value of $S(x)$. We call the limit $S(x)$ the **pointwise** limit of $S_n(x)$ as $n \to \infty$. In other words, the partial sums $S_n(x)$ converges **pointwise** to $S(x)$. There are other types of convergence as we shall see later on.

1.5 Fourier Series

Let $u \in L^1(-\pi,\pi)$ and define

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\,dx.$$

The series

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + b_k \sin kx$$

is called the **real Fourier series** of the function u. The real constants a_k and b_k (if u is real) are called the **Fourier coefficients** of u.
We will write that

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k\cos kx + b_k \sin kx\right).$$

Why not equality? Well, there's a couple of problems here.

(i) For a given $x \in [-\pi,\pi]$, does $S(x)$ exists? That is, does the series converge?

(ii) If $S(x)$ does exists, is it true that $S(x) = u(x)$?

(iii) If we consider $u \in L^1(-\pi,\pi)$, what does even $u(x)$ mean?

(iv) Suppose that $S(x)$ does exist and that $S(x) = u(x)$, in what way do we expect the partial sums to converge?

So when we write that $u(x) \sim S(x)$ we mean that $S(x)$ is the expression that we obtain from u when calculating the Fourier series. We will show that most of the questions above will have an answer with this meaning.

Example

Suppose that $u(x) = \operatorname{sgn}(x)$ for $x \in [-\pi,\pi]$, where $\operatorname{sgn}(x) = -1$ when $x < 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = 1$ when $x > 0$. Find the Fourier series of u.

Solution. We consider the periodic extension of u. The Fourier coefficients can be calculated as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos(0\cdot x)\,dx = \frac{1}{\pi}(-1+1) = 0,$$

and for $k \geq 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos kx\,dx + \int_{0}^{\pi}\cos kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[-\frac{\sin kx}{k}\right]_{-\pi}^{0} + \left[\frac{\sin kx}{k}\right]_{0}^{\pi}\right) \\ &= \frac{1}{\pi}\left(-\frac{-\sin(-k\pi)}{k} + \frac{\sin k\pi}{k}\right) = \frac{2\sin k\pi}{k} = 0, \end{aligned}$$

and finally for $k \geq 1$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin kx\,dx + \int_{0}^{\pi} \sin kx\,dx\right) \\ &= \frac{1}{\pi}\left(\left[\frac{\cos kx}{k}\right]_{-\pi}^{0} + \left[\frac{-\cos kx}{k}\right]_{0}^{\pi}\right) \\ &= \frac{1}{\pi}\left(\frac{1}{k} - \frac{\cos k\pi}{k} + \frac{-\cos k\pi}{k} + \frac{1}{k}\right) = \frac{2-2\cos k\pi}{k\pi} = \frac{2(1-(-1)^k)}{k\pi}. \end{aligned}$$

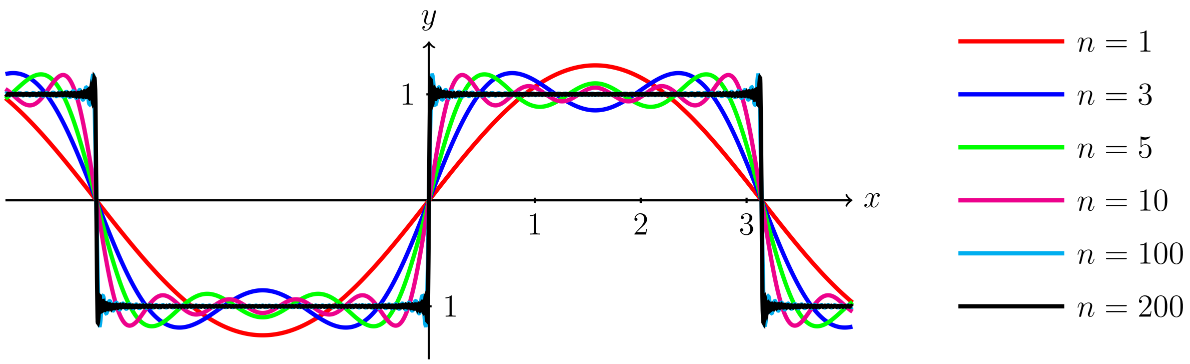
Hence

$$u(x) \sim \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k}\cdot \sin kx.$$

Now, a reasonable question is: "does this series converge?" Since, if k is odd,

$$\left|\frac{1-(-1)^k}{k}\cdot \sin kx\right| = \frac{2}{k}|\sin kx|,$$

it is *not* absolutely convergent. The series passes the divergence test, but that only means we cannot conclude that it is divergent. It might be tempting to think of Leibniz, but this series is not alternating (we might find some values for x but not in general). So we don't know if the series converges or diverges for just about any value of x. Don't worry, we'll get to this. In fact, this series is actually convergent to $u(x)$ for every x, but we have no idea why at this point. Summing the first n terms, we find the graphs below. This indicates that the sum indeed converges to the desired function, but there's some "squiggly" stuff going on around the jump points. We'll get back to that as well.



1.5.1 Complex Fourier series

So when examining the example in the previous section, we see that the same type of calculations are repeated for cos and sin. Considering that we've seen this phenomenon previously in analysis courses, might we consider a complex form instead and obtain both results at once? The answer is yes.

Similarly to above, let $u \in L^1(-\pi,\pi)$ and define

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx.$$

The series

$$u(x) \sim S(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is called the **complex Fourier series** of u and c_k are the **complex Fourier coefficients** of u. In this case, we define the partial sums $S_n(x) = \sum_{k=-n}^{n} u_k(x)$ so that we sum symmetrically around $n = 0$. Note that this gives a different type of convergence than if we were to have two different limits.
So how does this connect to the real Fourier series? Well, if we recall Euler's formulas, we have

$$e^{ikx} = \cos kx + i \sin kx.$$

Thus we see that

$$c_{\pm k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\big(\cos(\pm kx) - i\sin(\pm kx)\big)\,dx = \frac{1}{2}\left(a_k \mp ib_k\right),$$

and therefore, for $k > 0$,

$$\begin{aligned} c_k e^{ikx} + c_{-k}e^{-ikx} &= \frac{1}{2}\left(a_k - ib_k\right)\left(\cos kx + i\sin kx\right) + \frac{1}{2}\left(a_k + ib_k\right)\left(\cos kx - i\sin kx\right) \\ &= \frac{1}{2}\left(2a_k \cos kx + 2b_k \sin kx\right) = a_k \cos kx + b_k \sin kx. \end{aligned}$$

Hence the two types of partial sums (the real and the complex) are equal, so converges to the same thing if convergent (which they are at the same time). The condition that $u \in L^1(-\pi, \pi)$ is natural in the sense that this will ensure that the Fourier coefficients exist as absolutely convergent integrals:

$$|c_k| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|\left|e^{-ikx}\right|\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|\,dx.$$

When dealing with the complex Fourier coefficients, there are several different notations that are quite common. We might use these at certain points:

$$c_k = \widehat{u}_k = \widehat{u}[k], \quad k \in \mathbf{Z}.$$

So which representation is the best? That depends on the situation. The real series is clearly real valued (if u is real valued), which might be nice to see when working with real functions. However, the complex series is more compact and you can do more calculations at the same time. So the choice is basically yours, but be aware that you need to be able to handle both variants to pass the course. There's also some slight differences in function spaces used, so be careful which series you work with. In these notes, most things will be carried out using the complex form, whereas the book does most things with the real form. So there. You can choose yourself.

1.6 Frequency Domain?

Another thing that's straightforward with the complex notation is that we can plot some graphs that describe the "frequency content" of a periodic function. Consider the function

$$u(x) = 1 + 3\cos x - 2\cos 2x + 6\cos 4x + 4\cos 7x.$$

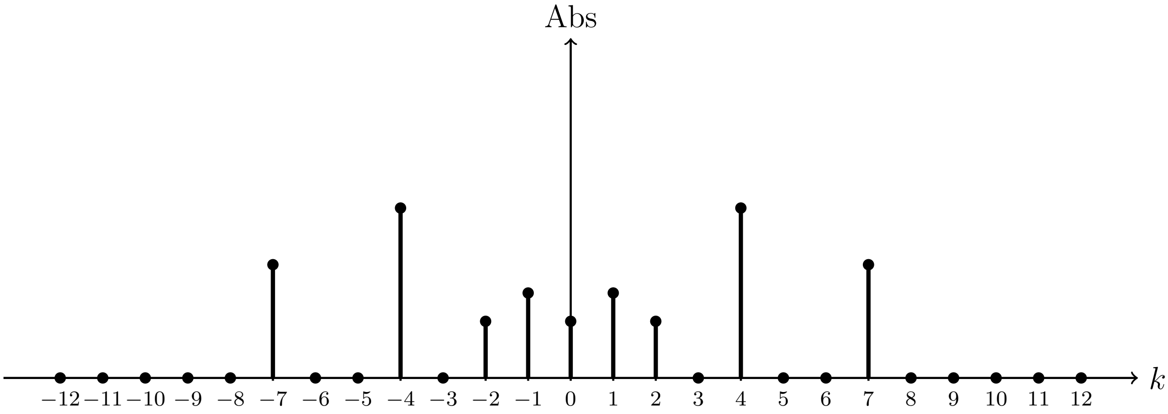
Using Euler's formulas, we can rewrite this as

$$u(x) = 1 + \frac{3}{2}e^{ix} + \frac{3}{2}e^{-ix} - e^{i2x} - e^{-i2x} + 3e^{i4x} + 3e^{-i4x} + 2e^{i7x} + 2e^{-i7x}.$$

This is the Fourier series for $u(x)$, although this is not exactly clear at the moment since we haven't shown any results regarding the uniqueness. As an exercise, try to use this representation to calculate the Fourier coefficients. You'll find that

$$c_0 = 1, \quad c_{\pm 1} = \frac{3}{2}, \quad c_{\pm 2} = -1, \quad c_{\pm 4} = 3, \quad \text{and } c_{\pm 7} = 2.$$

All other $c_k = 0$. What we usually do is draw the magnitude $|c_k|$ of the coefficients c_k (remember that they might be complex as well as negative). For this example, this would look like the graph below.



From this graph we see what frequencies are needed to represent a periodic function. This type of plot will become more interesting when we consider the Fourier transform instead. If we have used a real Fourier series, the magnitude is given by $\sqrt{a^2+b^2}$ and we only plot for nonnegative k (why?).

For something a little messier, let's consider the following.

Example

Let $u(x) = \cos\left(\frac{x}{2}\right)$, $-\pi \leq x \leq \pi$, and find the Fourier series of u. Draw a magnitude plot.

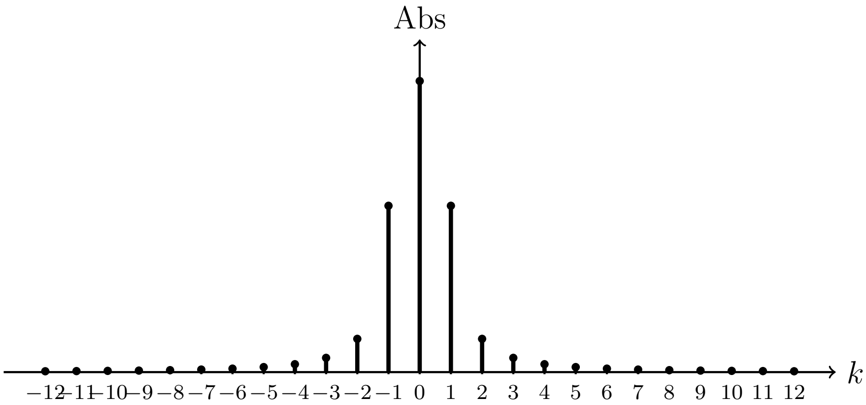
Solution. We need the Fourier coefficients, so

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right)\, dx = \frac{2}{\pi}$$

and for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\cos\left(\frac{x}{2}\right)\, dx = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{-ikx+ix/2}+e^{-ikx-ix/2}\right)\, dx \\ &= \frac{1}{4\pi}\left[\frac{e^{-ikx+ix/2}}{i(-k+1/2)}+\frac{e^{-ikx-ix/2}}{i(-k-1/2)}\right]_{-\pi}^{\pi} = \frac{(-1)^k}{2\pi}\left(\frac{1}{-k+1/2}-\frac{1}{-k-1/2}\right) \\ &= \frac{(-1)^{k+1}}{2\pi}\left(\frac{1}{(-k+1/2)(-k-1/2)}\right) = \frac{4(-1)^{k+1}}{2\pi(4k^2-1)}. \end{aligned}$$

We note that $c_k = \frac{4(-1)^{k+1}}{2\pi(4k^2-1)}$ for all $k \in \mathbf{Z}$, so $u(x) \sim \sum_{k=-\infty}^{\infty} \frac{4(-1)^{k+1}}{2\pi(4k^2-1)}\, e^{ikx}$.



We note that $c_k \neq 0$ for every $k \in \mathbf{Z}$ (they do tend to zero quite fast however), unlike the previous example where only certain values of k where nonzero. If only a finite number of c_k are nonzero, this means that the function is a trigonometric polynomial that is periodic with period 2π. While $\cos(x/2)$ is periodic, it is not periodic with period 2π. This is an important distinction.

1.7 Even/Odd Functions

Recall that a function u is even if $u(-x) = u(x)$ and odd if $u(-x) = -u(x)$. The most common examples being that $u(x) = \cos x$ is even and $u(x) = \sin x$ is odd. For functions that espouse these additional symmetries, we can make some simplifications to the Fourier calculations.

Theorem.

(i) If u is even, then $b_k = 0$ for $k = 1, 2, 3, \ldots$.

(ii) If u is odd, then $a_k = 0$ for $k = 1, 2, 3, \ldots$.

Proof. If u is even, then the product $u(x)\sin kx$ is odd for $k = 1, 2, 3, \ldots$. Hence

$$\int_{-\pi}^{\pi} u(x)\sin kx\,dx = 0,$$

so $b_k = 0$. Similarly, if u is odd, then $u(x)\cos kx$ is odd for $k = 1, 2, 3, \ldots$ which implies that $a_k = 0$.

Example

Find the Fourier series for $u(x) = x^2$, $x \in [-\pi, \pi]$.

Solution. First alternative: the real form. Since u is even, we know that $b_k = 0$. This means that we'll obtain a pure cosine-series. With this in mind, we calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

and

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\,dx = /\ x^2\cos kx \text{ even } / = \frac{2}{\pi}\int_0^{\pi} x^2\cos kx\,dx \\ &= /\ \text{I.B.P.}\ / = \frac{2}{\pi}\left(\left[\frac{x^2\sin kx}{k} + \frac{2x\cos kx}{k^2}\right]_0^{\pi} - \frac{2}{k^2}\int_0^{\pi}\cos kx\,dx\right) \\ &= \frac{2}{\pi}\left(\frac{2\pi\cos(\pi k)}{k^2}\right) = \frac{4(-1)^k}{k^2}. \end{aligned}$$

Alternative two: the complex form. Ignoring for a moment that we know that u is even, we could just do the calculation for the complex Fourier coefficients without using any additional information. Indeed,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

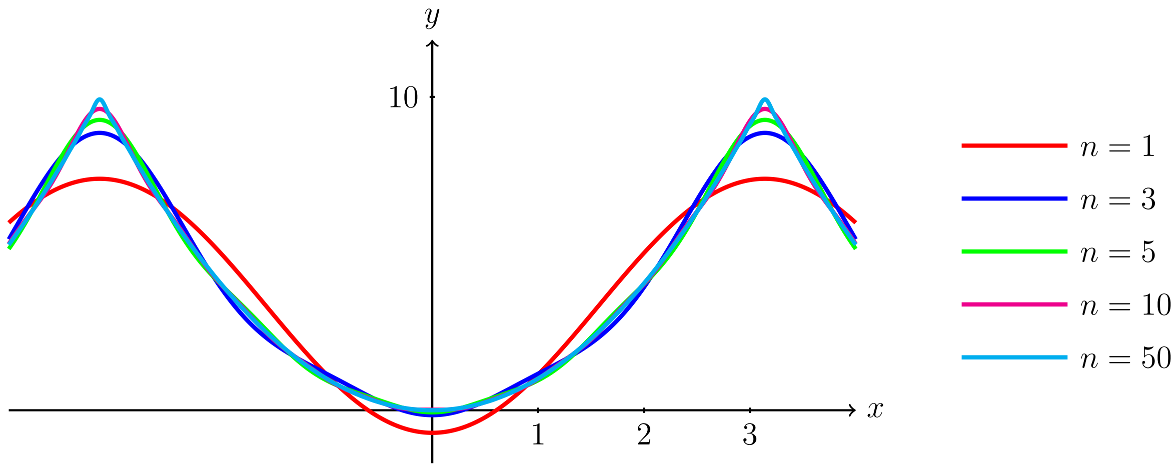
and for $k \neq 0$:

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-ikx}\,dx = /\ \text{I.B.P.}\ / = \frac{1}{2\pi}\left(\left[-\frac{1}{ik}x^2e^{-ikx} + \frac{2x}{k^2}e^{-ikx}\right]_{-\pi}^{\pi} - \frac{2}{k^2}\int_{-\pi}^{\pi} e^{-ikx}\,dx\right) \\ &= \frac{1}{2\pi}\left(\frac{4\pi(-1)^k}{k^2}\right) \end{aligned}$$

Due to the symmetry $c_{-k} = c_k$, we obtain the same pure cosine series as before.
So we have shown that

$$u(x) \sim \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2}\cos kx.$$

We note that the series is actually absolutely convergent, so we do know that it converges. Is it equal to x^2 for $x \in [-\pi, \pi]$? At this point, we do not know. Obviously there's still some theory that we're missing. Drawing the graphs for the partial sums, we find that the Fourier series seems to converge to x^2 (periodically extended). Note that there seems to be nothing of that squiggly behavior we saw when drawing the partial sums for $\mathrm{sgn}(x)$. Why not?



1.7.1 Even/Odd Extensions

Suppose that we have a function $u\colon [0,\pi] \to \mathbf{C}$. We define the even extension u_e of u by

$$u_e(x) = \begin{cases} u(x), & 0 \le x \le \pi, \\ u(-x), & -\pi \le x < 0, \end{cases}$$

and the odd extension u_o of u by

$$u_o(x) = \begin{cases} u(x), & 0 < x \leq \pi, \\ 0, & x = 0, \\ -u(-x), & -\pi \leq x < 0, \end{cases}$$

So note that we only have a function defined on half the interval $[-\pi, \pi]$ and that we extend this to the other half. Since we now obtain an odd or even function (depending on choice), we find that the Fourier series will contain only sine or cosine terms. We call this the **sine series** or **cosine series** for a function $u \in L^2(0, \pi)$.

1.8 What if $T \neq 2\pi$?

As stated earlier, it's not a problem to use functions with a different period than 2π. For this purpose, if u is a T-periodic function, we define

$$\Omega = \frac{2\pi}{T}.$$

The real Fourier series of u is then given by

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\Omega x + b_k \sin k\Omega x,$$

where

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\cos k\Omega x\,dx \quad \text{and} \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\sin k\Omega x\,dx.$$

The complex series is given by

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega x}, \quad \text{where } c_k = \frac{1}{T}\int_{-T/2}^{T/2} u(x)e^{-ik\Omega x}\,dx.$$

Example

Find the Fourier series of $u(x) = |x|$, $-1 \leq x \leq 1$.

Solution. We consider the periodic extension of u with the period $T = 2$. Then $\Omega = 2\pi/2 = \pi$ and for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2}\int_{-1}^{1} |x|e^{-ik\pi x}\,dx = \frac{1}{2}\int_{-1}^{0} -xe^{-ik\pi x}\,dx + \frac{1}{2}\int_{0}^{1} xe^{-ik\pi x}\,dx \\ &= \frac{1}{2}\left(\left[\frac{-xe^{-ik\pi x}}{-ik\pi}\right]_{-1}^{0} + \int_{-1}^{0}\frac{e^{-ik\pi x}}{-ik\pi}\,dx\right) + \frac{1}{2}\left(\left[\frac{xe^{-ik\pi x}}{-ik\pi}\right]_{0}^{1} - \int_{0}^{1}\frac{e^{-ik\pi x}}{-ik\pi}\,dx\right) \\ &= \frac{1}{2}\left(\frac{e^{ik\pi}}{ik\pi} + \left[\frac{e^{-ik\pi x}}{-k^2\pi^2}\right]_{-1}^{0}\right) + \frac{1}{2}\left(-\frac{e^{-ik\pi}}{ik\pi} - \left[\frac{e^{-ik\pi x}}{-k^2\pi^2}\right]_{0}^{1}\right) \\ &= \frac{1}{2}\left(-\frac{1}{k^2\pi^2} + \frac{e^{ik\pi}}{k^2\pi^2} + \frac{e^{-ik\pi}}{k^2\pi^2} - \frac{1}{k^2\pi^2}\right) = \frac{(-1)^k - 1}{k^2\pi^2}. \end{aligned}$$

For $k = 0$,

$$c_0 = \frac{1}{2}\int_{-1}^{1} |x|\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

Hence

$$u(x) \sim \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k - 1}{k^2\pi^2} e^{ik\pi x} = \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2\pi^2} \cos((2k+1)\pi x),$$

where the last expression follows from Euler's formulas and the fact that $c_{-k} = c_k$ and $c_{2k} = 0$ for $k \in \mathbf{Z}$ with $k \neq 0$.

Chapter 2

Linear Algebra, Infinite Dimensional Spaces and Functional Analysis

"You have no respect for logic. I have no respect for those who have no respect for logic."
—Julius Benedict

2.1 Remember Linear algebra? Finite Dimensional Spaces

Let V be a **linear space** (sometimes we say vector space) over the complex (sometimes real) numbers. We recall some definitions from linear algebra. Elements of a linear space can be added and multiplied by constants and still belong to the linear space:

$$u, v \in V \quad \Rightarrow \quad \alpha u + \beta v \in V, \quad \alpha, \beta \in \mathbf{C} \text{ (or } \mathbf{R}).$$

The operations addition and multiplication by constant behaves like we expect (associative, distributive and commutative). Multiplication of vectors is not defined in general, but as we shall see we can define different useful products in many cases.

Linear combination

Definition. Let $u_1, u_2, \dots, u_n \in V$. We call

$$u = \sum_{k=1}^{n} \alpha_k u_k = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

a **linear combination**. If

$$\sum_{k=1}^{n} \alpha_k u_k = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

we say that $u_1, u_2, \dots, u_n$ are **linearly independent**. The **linear span** $\text{span}\{u_1, u_2, \dots, u_n\}$ of the vectors $u_1, u_2, \dots, u_n$ is defined as the set of all linear combinations of these vectors (which is a linear space).

You've seen plenty of linear spaces before. One such example is the euclidian space $\mathbf{R}^n$ consisting of elements $(x_1, x_2, \dots, x_n)$, where $x_i \in \mathbf{R}$. Recall also that you've seen linear spaces that consisted of polynomials. The fact that our definitions is general enough to cover many cases will prove to be very fruitful.

Basis

Definition. A subset $\{v_1, v_2, \ldots, v_n\} \subset V$ of linearly independent vectors is called a **base** for V if $V = \text{span}\{v_1, v_2, \ldots, v_n\}$ (meaning that every vector $v \in V$ can be expressed uniquely as a linear combination of the elements $v_1, v_2, \ldots, v_n$). The non-negative integer n is called the **dimension** of V: $\dim(V) = n$.

In general, we do not wish to restrict ourselves to finite dimensions or vectors of complex numbers.

2.1.1 Sequences

We denote a sequence $u_1, u_2, u_3, \ldots$ (or $u_1, u_2, \ldots, u_n$ if it is a finite sequence) of elements of a linear space V by $(u_k)_{k=1}^{\infty}$ ($(u_k)_{k=1}^{n}$). If there's no risk of misunderstanding, we might just say "the sequence u_n."

As an example, consider the sequence $u_n = x + \dfrac{1}{n}$ in $\mathbf{R}$. That means that $u_1 = x + 1$, $u_2 = x + 1/2$, $u_3 = x + 1/3$, and so on. We see that as $n \to \infty$, clearly $u_n \to x$. In other words, the sequence u_n *converges* to x. This feels natural in this setting, but we will generalize this to have meaning for other linear spaces than $\mathbf{R}$.

2.2 Normed Linear Spaces

To measure distances between elements in a linear space (or "lengths" of elements), we define the abstract notion of a **norm** on a linear space (in the cases where this is allowed).

Norm

Definition. A normed linear space is a linear space V endowed with a norm $\|\cdot\|$ that assigns a non-negative number to each element in V in a way such that

(i) $\|u\| \geq 0$ for every $u \in V$,

(ii) $\|\alpha u\| = |\alpha|\|u\|$ for $u \in V$ and every constant α,

(iii) $\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in V$.

We note that in linear algebra, we typically used the norm $|\cdot|$ on the euclidean space $\mathbf{R}^n$ (or $\mathbf{C}^n$). We will use different types of norms in this course since we will be dealing with more complex spaces.

An element e in V with length 1, that is, $\|e\| = 1$, is called a **unit vector**.

Some examples of normed spaces

(i) The space $\mathbf{R}^n$ with the norm $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

(ii) The space $\mathbf{R}^n$ with the norm $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.

The first example is obviously already something you're familiar with. It is also an example of something we will call an inner product space below. The second example is a bit different. In some sense equivalent, but the norms yield different values for the same vector. Try to prove that the second one satisfies all the requirements for a norm.

The space of continuous functions with sup-norm

The space $C[a, b]$ consisting of continuous functions on the closed interval $[a, b]$ endowed with the norm

$$\|f\|_{C[a,b]} = \max_{a \le t \le b} |f(t)|, \quad f \in C[a, b].$$

Sequence spaces

The space l^1 consisting of all sequences $(x_1, x_2, x_3, \ldots)$ such that the norm

$$\|x\|_{l^1} = \sum_{k=1}^{\infty} |x_k| < \infty.$$

We might also consider the space l^p for $1 \le p < \infty$ with the norm

$$\|x\|_{l^p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

The space of absolutely integrable functions

The space $L^1(\mathbf{R})$ of all integrable (on $\mathbf{R}$) functions with the norm

$$\|f\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

In other words, all functions that are absolutely integrable on $\mathbf{R}$. Note here that there's an army of dogs buried here. Indeed, the integral is not in the sense we're used to but rather in the form of the Lebesgue integral. We will not get stuck at this point, but it might be good to know.

Exercise: Prove that the spaces above are normed linear spaces. Do you see any useful ways to consider some "multiplication" of vectors?
We see that an underlying linear space (like $\mathbf{R}^n$) might be endowed with different norms. This is true in general, and changing norms usually changes the results (at least for infinite dimensional spaces).

2.3 Convergence in Normed Spaces

Let $u_1, u_2, \ldots$ be a sequence in a normed space V. We say that $u_n \to u$ for some $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$. This is called **strong convergence** or **convergence in norm**. Note that we assumed above that the element u belonged to V. This may not be the case for every convergent sequence.

Cauchy sequence

Definition. We call a sequence $(u_k)_{k=1}^{\infty}$ in V a Cauchy sequence if for every $\epsilon > 0$ there exists an integer N so that

$$\|u_n - u_m\| \leq \epsilon \quad \text{for } n, m \geq N.$$

Complete space

Definition. If every Cauchy sequence u_n in V converges to an element in V, say $u_n \to u \in V$, we call V complete.

Closed space

Definition. If every convergent sequence u_n in V converges to an element $u \in V$, that is

$$u_n \to u \quad \Rightarrow \quad u \in V,$$

we call V (sequentially) closed.

Note that a complete space is closed but that the reverse is not necessarily true for general spaces (some metric spaces for example).
For this course, we will mainly study the space E which consists of piecewise continuous functions. This will ensure that some things are easy, but unfortunately the space E with the norms and inner products we are interested in will not be complete nor closed. This will not be a big problem for us, but it's worth mentioning if we wish to do Fourier analysis in a more generalized setting.
Analogously with real analysis, we can define continuous mappings on normed spaces.

Continuity in normed spaces

Definition. Let V and W be normed spaces. A function $u\colon V \to W$ is said to be continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$x, y \in V, \ \|x - y\|_V < \delta \quad \Rightarrow \quad \|u(x) - u(y)\|_W < \epsilon.$$

2.3.1 Series in Normed Spaces

Let $u_1, u_2, u_3, \ldots$ be a sequence in V. How do we interpret an expression of the form

$$S = \sum_{k=1}^{\infty} u_k, \tag{2.1}$$

that is, what does an infinite sum of elements in V mean? We define the partial sums by

$$S_n = \sum_{k=1}^{n} u_k, \quad n = 1, 2, 3, \ldots$$

If S_n converges to some $S \in V$ *in norm*, that is,

$$\lim_{n\to\infty} \left\| S - \sum_{k=1}^{n} u_k \right\| = 0,$$

then we write that (2.1) is convergent. Notice that this does *not* mean that

$$\sum_{k=1}^{\infty} \|u_k\| < \infty.$$

If this second series of real numbers is convergent, we call (2.1) **absolutely convergent** (compare with what we did in TATA42). Note also that an absolutely convergent series is convergent in the sense above (why?).

2.4 Inner Product Spaces

A norm is not enough to define a suitable geometry for our purposes, so we will usually work with inner product spaces instead.

Inner product

Definition. An inner product $\langle \cdot\,, \ \cdot \rangle$ on a vector space V is a complex valued (sometimes real) function on $V \times V$ such that

(i) $\langle u, \ v \rangle = \overline{\langle v, \ u \rangle}$

(ii) $\langle u + v, \ w \rangle = \langle u, \ w \rangle + \langle v, \ w \rangle$

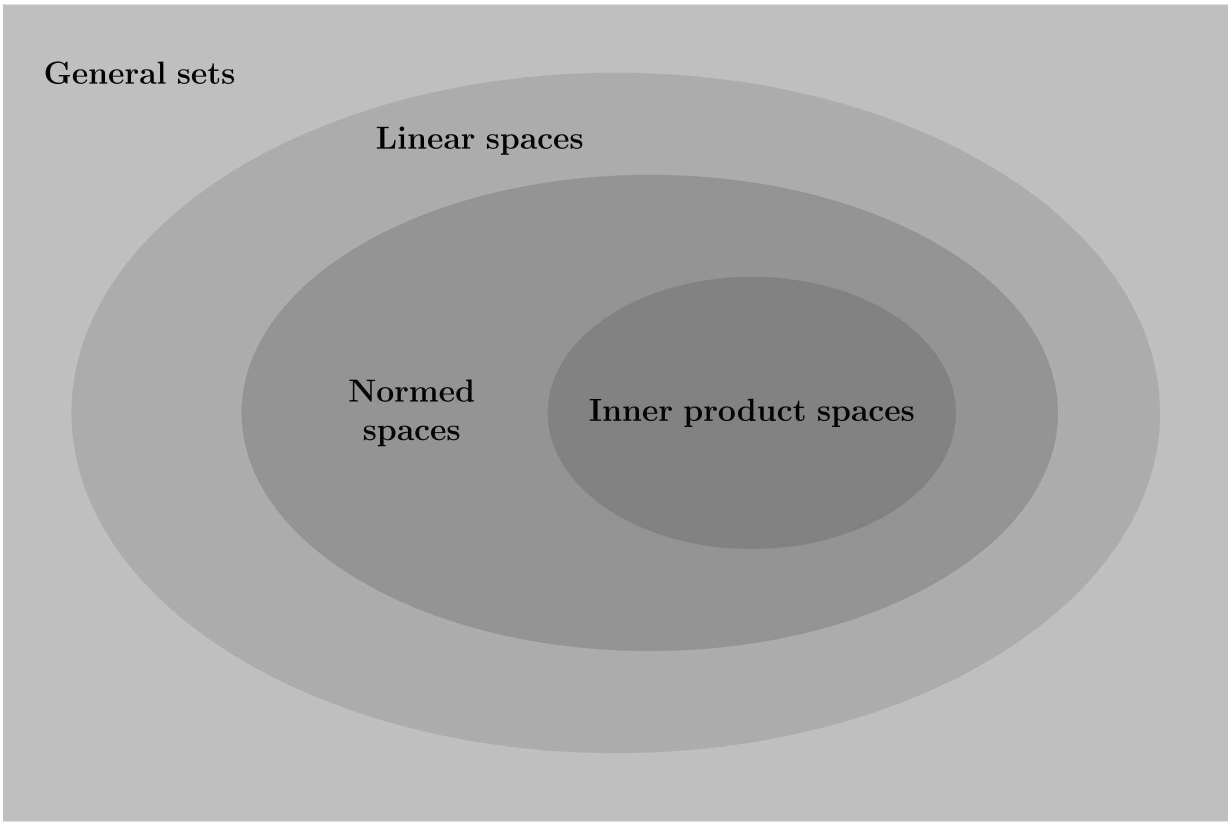
(iii) $\langle \alpha u, \ v \rangle = \alpha \langle u, \ v \rangle$

(iv) $\langle u, \ u \rangle \geq 0$

(v) $\langle u, \ u \rangle = 0$ if and only if $u = 0$.

Note that (i) and (ii) implies that $\langle u, \ v + w \rangle = \langle u, \ v \rangle + \langle u, \ w \rangle$ and that (i) and (iii) implies that $\langle u, \ \alpha v \rangle = \overline{\alpha} \langle u, \ v \rangle$.
In an inner product space, we use $\|u\| = \sqrt{\langle u, \ u \rangle}$ as the norm. Why is this a norm? We'll get to that.

Notice that if we're given a linear space of functions, there's an infinite number of different inner products on this space that provides the same *geometry*. Suppose that $\langle u, \ v \rangle$ is an inner product. Then $\alpha \langle u, \ v \rangle$ is also an inner product for any $\alpha > 0$.



The inner product space $\mathbf{C}^n$

Definition. The space $\mathbf{C}^n$ consisting of n-tuples $(z_1, z_2, \ldots, z_n)$ with

$$\langle z,\ w\rangle = \sum_{k=1}^{n} z_k \overline{w_k}, \quad z, w \in \mathbf{C}^n,$$

is an inner product space.

The inner product space l^2

Definition. The space l^2 consisting of all sequences $(x_1, x_2, x_3, \ldots)$ of complex numbers such that the norm

$$\|x\|_{l^2} = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2} < \infty.$$

This is an inner product space if

$$\langle x,\ y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$

The inner product space $L^2(a, b)$

Definition. The space $L^2(a, b)$ consists of all "square integrable" functions with the inner product

$$\langle f,\ g\rangle = \int_a^b f(t)\overline{g(t)}\,dt.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed.

Why not the same examples as for the normed spaces? The simple answer is that most of those examples are *not* inner product spaces. The last two examples above are very important and the fact that it's the number 2 is not random and this is actually the only choice for when $L^p(a, b)$, which consists of functions for which

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} < \infty$$

are inner product spaces. Again, we also note that the integrals above are more general than what we've seen earlier but if the function f is nice enough the value will coincide with the (generalized) Riemann integral.

Orthogonality

Definition. If $u, v \in V$ and V is an inner product space, we say that u and v are orthogonal if $\langle u,\ v\rangle = 0$. We denote this by $u \perp v$.

A sequence u_n is called **pairwise orthogonal** if $\langle u_i,\ u_j\rangle = 0$ for every $i \neq j$. For sequences of this type, we have the generalized Pythagorean theorem.

Theorem. If $u_1, u_2, \ldots, u_n$ are pairwise orthogonal, then

$$\|u_1 + u_2 + \cdots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2.$$

The Cauchy-Schwarz inequality

Theorem. If $u, v \in V$ and V is an inner product space, then $|\langle u,\ v\rangle| \leq \|u\|\|v\|$.

Proof. Assume that $v \neq 0$ (the inequality is trivial if $v = 0$) and define $\lambda = \langle u,\ v\rangle / \|v\|^2$. Then

$$\begin{aligned}\|u - \lambda v\|^2 &= \langle u - \lambda v,\ u - \lambda v\rangle = \|u\|^2 - \overline{\lambda}\langle u,\ v\rangle - \lambda\langle v,\ u\rangle + |\lambda|^2\|v\|^2 \\ &= \|u\|^2 - \overline{\lambda}\langle u,\ v\rangle - \lambda\overline{\langle u,\ v\rangle} + |\lambda|^2\|v\|^2 \\ &= \|u\|^2 - \frac{|\langle u,\ v\rangle|^2}{\|v\|^2} - \frac{|\langle u,\ v\rangle|^2}{\|v\|^2} + \frac{|\langle u,\ v\rangle|^2}{\|v\|^2} = \|u\|^2 - \frac{|\langle u,\ v\rangle|^2}{\|v\|^2}\end{aligned}$$

so

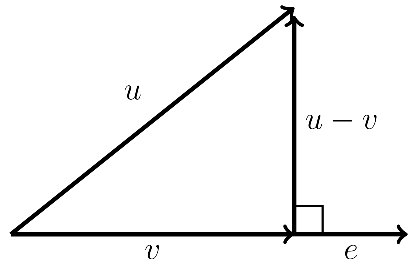
$$0 \leq \|u - \lambda v\|^2 = \|u\|^2 - \frac{|\langle u,\ v\rangle|^2}{\|v\|^2} \quad \Leftrightarrow \quad |\langle u,\ v\rangle|^2 \leq \|u\|^2\|v\|^2,$$

which implies the inequality. $\square$

2.5 Orthogonal Projection

Let $e \in V$ with $\|e\| = 1$. For $u \in V$, we define the **orthogonal projection** v of u on e by $v = \langle u,\ e\rangle\, e$. This is reasonable since $u - v \perp e$:

$$\langle u - v,\ e\rangle = \langle u,\ e\rangle - \langle v,\ e\rangle = \langle u,\ e\rangle - \langle u,\ e\rangle \langle e,\ e\rangle = 0.$$



Note that

$$\begin{aligned}\|u\|^2 &= \|u - v + v\|^2 \\ &= \|u - v\|^2 + \|v\|^2 \\ &= \|u - v\|^2 + |\langle u,\ e\rangle|^2.\end{aligned}$$

ON system

Definition. Let V be an inner product space. We call

(i) $\{e_1, e_2, \ldots, e_n\} \subset V$,

(ii) or $\{e_1, e_2, \ldots\} \subset V$,

an ON system in V if $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$ for all i.

We do *not* assume that V is finite dimensional and that n is the dimension, and we do *not* assume that the ON system consists of finitely many elements.
If the ON system is finite, consider $W = \text{span}\{e_1, e_2, \ldots, e_n\} \subset V$. We define the orthogonal projection Pv of a vector $v \in V$ onto the linear space W by

$$Pv = \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k.$$

If $v \in W$, then clearly $Pv = v$. If $v \notin W$, then Pv is the vector that minimizes $\|v - Pv\|$. Note that this happens if $v - Pv \perp W$ (meaning perpendicular to every vector in W). We also note that

$$\|v\|^2 = \|v - Pv\|^2 + \sum_{k=1}^{n} |\langle v,\ e_k\rangle|^2$$

These facts are well-known from linear algebra.
If the ON system is infinite, let

$$P_n v = \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k, \quad v \in V,\ n = 1, 2, 3, \ldots$$

Each $P_n v$ is the projection on a specific n-dimensional subspace of V (the order of the elements in the ON system is fixed). Let us state the famous Bessel's inequality.

Bessel's inequality

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \dots\}$ be an ON system in V. Then

$$\sum_{k=1}^{\infty} |\langle v,\ e_k \rangle|^2 \leq \|v\|^2.$$

Since $\|v\| < \infty$ for every $v \in V$, this inequality proves that the series in the left-hand side converges. A direct consequence of this is the Riemann-Lebesgue lemma.

The Riemann-Lebesgue Lemma

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \dots\}$ be an ON system in V. Then

$$\lim_{n \to \infty} \langle v,\ e_n \rangle = 0.$$

2.5.1 The Infinite Dimensional Case

If $\dim(V) = n$ and our ON system has n elements, then we know that we can always represent $v \in V$ as $v = \sum_{k=1}^{n} \langle v,\ e_k \rangle e_k$ (standard linear algebra). What happens if $\dim(V) = \infty$? When can we expect that an ON systems allows for something similar?

Closed ON systems

Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \dots\} \subset V$ **closed** if for every $v \in V$ and every $\epsilon > 0$, there exists a sequence $c_1, c_2, \dots, c_n$ of constants such that

$$\left\| v - \sum_{k=1}^{n} c_k e_k \right\| < \epsilon. \tag{2.2}$$

How do we typically find numbers c_k that work (they're not unique)? One answer comes in the form of orthogonal projections.

Fourier coefficients

Definition. For a given ON system, the complex numbers $\langle v,\ e_k \rangle$, $k = 1, 2, \dots$, are called the **generalized Fourier coefficients** of v.

We define the operator P_n that projects a vector onto the linear space spanned by $\{e_1, e_2, \dots, e_n\}$ by

$$P_n v = \sum_{k=1}^{n} \langle v,\ e_k \rangle e_k, \quad v \in V.$$

We now note that the choice $c_k = \langle v,\ e_k\rangle$ is the choice that minimizes the left-hand side in (2.2). Indeed, suppose that $u = \sum_{k=1}^{n} c_k e_k$ for some constants c_k. Then

$$\begin{aligned}\|v-u\|^2 &= \|v - P_n v + P_n v - u\|^2 = /\ (v - P_n v) \perp (P_n v - u)\ / = \|v - P_n v\|^2 + \|P_n v - u\|^2 \\ &= \|v - P_n v\|^2 + \left\|\sum_{k=1}^{n}(\langle v,\ e_k\rangle - c_k)e_k\right\|^2 \\ &= \|v - P_n v\|^2 + \sum_{k=1}^{n} |\,\langle v,\ e_k\rangle - c_k|^2,\end{aligned}$$

so obviously $c_k = \langle v,\ e_k\rangle$ is the unique choice that minimizes $\|v-u\|$. In other words, $u = P_n v$ is the only element that minimizes $\|v-u\|$.
Because of this, one can reformulate (equivalently) the definition of a closed ON system as follows.

Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \ldots\} \subset V$ **closed** if for every $v \in V$

$$\lim_{n\to\infty} \left\| v - \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k \right\| = 0.$$

We note that in the case where the ON system is closed, we can strengthen Bessel's inequality (by replacing the inequality with equality) obtaining what is known as Parseval's identity (or Parseval's formula). As it turns out, the fact that Parseval's identity holds for an ON-system is equivalent to the fact that the ON-system is closed.

Theorem. Suppose that $W = \{e_1, e_2, \ldots\}$ is an ON system for the inner product space V. Then W is closed if and only if **Parseval's identity** holds:

$$\sum_{k=1}^{\infty} |\,\langle v,\ e_k\rangle\,|^2 = \|v\|^2$$

for every $v \in V$.

Proof. Let $v \in V$. Then

$$\|v\|^2 = \|v - P_n v\|^2 + \|P_n v\|^2$$

since $v - P_n v \perp P_n v$. Hence

$$\left\| v - \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k \right\|^2 = \|v\|^2 - \sum_{k=1}^{n} |\,\langle v,\ e_k\rangle\,|^2$$

and letting $n \to \infty$ in this equality, we see that closedness is equivalent with Parseval's identity holding.

Definition. An ON-system $\{e_1, e_2, \ldots\}$ in V is called complete if, for every $v \in V$,

$$\langle v,\ e_k\rangle = 0 \text{ for all } k = 1, 2, 3, \ldots \quad \Leftrightarrow \quad v = 0.$$

We realize that completeness is something we want if we wish to use an ON-system as a *basis* for V since this is needed to make representations in terms of linear combinations of basis vectors needs to be unique to avoid problems.

Generalized Parseval's identity

Theorem. Suppose that $\{e_1, e_2, e_3, \ldots\}$ is a closed infinite ON-system in V and let $u, v \in V$. If $a_k = \langle u,\ e_k\rangle$ and $b_k = \langle v,\ e_k\rangle$, then

$$\langle u,\ v\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

Proof. Since V is a complex inner product space, the following equality (usually known as the polarization identity) holds:

$$\langle u,\ v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right).$$

Since we have a closed ON-system, Parseval's formula holds, so it is clear that

$$\|u+v\|^2 = \sum_{k=1}^{\infty} |a_k + b_k|^2$$

since $\langle u+v,\ e_k\rangle = \langle u,\ e_k\rangle + \langle v,\ e_k\rangle = a_k + b_k$. Similarly, we obtain that

$$\|u-v\|^2 = \sum_{k=1}^{\infty} |a_k - b_k|^2, \quad \|u+iv\|^2 = \sum_{k=1}^{\infty} |a_k + ib_k|^2, \quad \|u-iv\|^2 = \sum_{k=1}^{\infty} |a_k - ib_k|^2.$$

Note also that (verify this directly)

$$a_k\overline{b_k} = \frac{1}{4}\left(|a_k+b_k|^2 - |a_k-b_k|^2 + i|a_k+ib_k|^2 - i|a_k-ib_k|^2\right),$$

so the identity in the theorem must hold.

2.6 Fourier Series?

So that brings us back to one of the main subjects of this course: Fourier series. Let's look at a particular inner product space.

2.6.1 The ON Systems

We consider the space $L^2(-\pi, \pi)$ consisting of square integrable functions $u\colon [-\pi, \pi] \to \mathbf{C}$:

$$\int_{-\pi}^{\pi} |u(x)|^2 \, dx < \infty.$$

We define the inner product on this space by

$$\langle u, \ v\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\overline{v(x)} \, dx.$$

Note that this infers that we have the norm

$$\|u\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x)|^2 \, dx \right)^{1/2},$$

which by definition is finite for $u \in L^2(-\pi, \pi)$. Let's consider two special orthonormal systems in this space.
The set of functions e^{ikx}, $k \in \mathbf{Z}$, is a closed orthonormal system in E with the inner product defined above. We consider E as a subspace of $L^2(-\pi, \pi)$. Clearly we have

$$\|e^{ikx}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ikx} \, dx = \frac{2\pi}{2\pi} = 1.$$

Similarly, if $k, l \in \mathbf{Z}$ and $k \neq l$, we have

$$\left\langle e^{ikx}, \ e^{ilx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)x} \, dx = 0$$

since $e^{i(k-l)x}$ is 2π-periodic. So this is an ON-system in E. The fact that it is closed is a more difficult argument so we'll get back to this on lecture 5. Note though, that E is a not closed in the more general space $L^2(-\pi, \pi)$, and not complete either. This is a disadvantage, but nothing that will cause too much problems for us.

The Real System

The set of functions $\frac{1}{\sqrt{2}}$, $\cos kx$, $k = 1, 1, 2, \ldots$, $\sin kx$, $k = 1, 2, 3, \ldots$, is a closed orthonormal system in E with the inner product

$$\langle u, \ v\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x)\overline{v(x)} \, dx.$$

Note that the *normalization constant* is different compared to the complex case (why do you think that is?). We should observe that these two systems are equivalent due to Euler's formulas.

2.7 The Space *E* as an Inner Product Space

So the question right now is *what do we really need?*

Most of the results we're going to see have a more general and complete (he he..) version, but we would need considerably more time to develop the necessary tools to attack these problems. So what we're going to do instead is to consider the space E with the inner product

$$\langle u,\ v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx, \quad u, v \in E. \tag{2.3}$$

This space has some serious drawbacks (the space E is not complete nor closed for example), but these problems are not crucial to what we're going to do.
First, let's verify that things work as expected. When we write E, we now mean the combination of the set E of piecewise continuous functions combined with the inner product defined by (2.3).

(i) *E is a linear space.* Obviously, if $u \in E$ and α is a constant, then αu has the same exception points as u (unless $\alpha = 0$) and the right- and lefthand limits will exist for $\alpha u(x)$. Let $u, v \in E$ and let $a_1, a_2, \ldots, a_n$ be the exceptions points of u and $b_1, b_2, \ldots, b_m$ be the exception points of v. Then $u + v$ has (at most) $m + n$ exception points. Indeed, if we sort the exception points as $c_1 < c_2 < \cdots < c_{n+m}$, then $u + v$ will be continuous on each $]c_i, c_{i+1}[$ and the right- and lefthand limits at the exception points will exist since either it is an exception point for u or v (potentially both), or it is a point of continuity for u or v. Therefore the limit of the sum exist.

(ii) *Equation* (2.3) *defines an inner product on E.* Most of the properties follow from the linearity of the integral. The fact that $\langle u,\ u\rangle = 0$ implies that $u = 0$ is clear since

$$\langle u,\ u\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = 0$$

so $u = 0$ is the only possible piecewise continuous function (if $u(x_0) \neq 0$ at some point then there is an interval $]x_0 - \delta, x_0 + \delta[$ where $|u(x)| > 0$ and so the Riemann integral will be strictly greater than zero).

2.7.1 Fourier Coefficients and the Riemann Lebesgue Lemma

So in general, we know that $\langle u,\ e_k\rangle \to 0$ as $k \to \infty$ if $\{e_1, e_2, \ldots\}$ is an ON system with respect to the inner product at hand (in our case (2.3)). This was a consequence of Bessel's inequality. In particular, this means that for $u \in E$, we have

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} u(x)e^{inx}\,dx = 0.$$

Note that this implies that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} u(x)\sin(nx)\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_{-\pi}^{\pi} u(x)\cos(nx)\,dx = 0.$$

So apparently these limits hold for all piecewise continuous functions. However, these identities are also true for $u \in L^1(-\pi, \pi)$ (this needs to be proved).

2.7.2 Bessel's Inequality Turned Parseval's Identity

Taking for granted that this ON system is closed (which is not clear at all at this point but we'll get back to that), we conclude by noting that Parsevals's identity looks like this:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(x)|^2\,dx = \sum_{k=-\infty}^{\infty}|c_k|^2,$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}u(x)e^{-ikx}\,dx, \quad k\in\mathbf{Z}.$$

The general form is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}u(x)\overline{v(x)}\,dx = \sum_{k=-\infty}^{\infty}c_k\overline{d_k},$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}u(x)e^{-ikx}\,dx \quad \text{and} \quad d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}v(x)e^{-ikx}\,dx, \quad k\in\mathbf{Z}.$$

2.8 Why is $\sqrt{\langle u,\ u\rangle}$ a Norm?

Let's define $\|u\| = \sqrt{\langle u,\ u\rangle}$ for $u\in V$. Then clearly $\|u\|\ge 0$ and $\|u\| = 0$ if and only if $u = 0$ (since this holds for the inner product). Furthermore, if $\alpha\in\mathbf{C}$ we have

$$\|\alpha u\| = \sqrt{\langle \alpha u,\ \alpha u\rangle} = \sqrt{\alpha\overline{\alpha}\,\langle u,\ u\rangle} = |\alpha|\sqrt{\langle u,\ u\rangle} = |\alpha|\|u\|.$$

To prove that $\|u+v\|\le\|u\|+\|v\|$, we note that

$$\|u+v\|^2 = \langle u+v,\ u+v\rangle = \|u\|^2+\|v\|^2+\langle u,\ v\rangle+\langle v,\ u\rangle = \|u\|^2+\|v\|^2+2\,\mathrm{Re}\,\langle u,\ v\rangle\,.$$

Since $\mathrm{Re}\,z\le|z|$ for any $z\in\mathbf{C}$ (why?), the Cauchy-Schwarz inequality implies that

$$2\,\mathrm{Re}\,\langle u,\ v\rangle \le 2|\,\langle u,\ v\rangle\,| \le 2\|u\|\|v\|.$$

Thus

$$\|u\|^2+\|v\|^2+2\,\mathrm{Re}\,\langle u,\ v\rangle \le \|u\|^2+\|v\|^2+2\|u\|\|v\| = (\|u\|+\|v\|)^2,$$

so

$$\|u+v\|^2 \le (\|u\|+\|v\|)^2,$$

which proves that the triangle inequality holds.

Chapter 3

Function Series and Convergence

"Here, stick around!"
—John Matrix

3.1 Pointwise Convergence

Let $u_1, u_2, u_3, \ldots$ be a sequence of functions $u_k : I \to \mathbf{C}$, where I is some set of real numbers. We've seen pointwise convergence earlier, but let's formulate it more rigorously.

Pointwise convergence

Definition. We say that $u_k \to u$ pointwise on I as $k \to \infty$ if

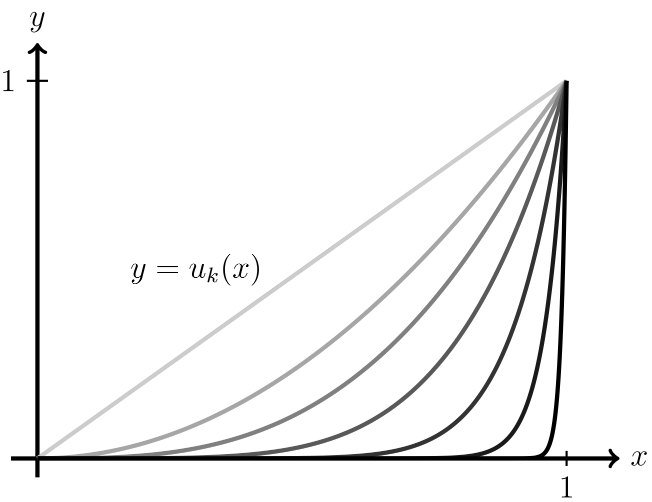
$$\lim_{k\to\infty} u_k(x) = u(x)$$

for every $x \in I$. We often refer to u as the *limiting function*.

Why would this not suffice? Let's consider an example.

Example

Let $u_k(x) = x^k$ if $0 \leq x \leq 1$, $k = 1, 2, 3, \ldots$. Then $u_k(x) \to 0$ for $0 \leq x < 1$ and $u_k(x) \to 1$ when $x = 1$. Clearly u_k is continuous on $[0, 1]$ for every k, but the *limiting function* is discontinuous at $x = 1$.



This is slightly troubling. The fact that certain properties hold for all elements in a sequence but not for the limiting element has caused more than one engineer to assume something dangerous. So can we require something more to ensure that, e.g., continuity is inherited? As we shall see, if the convergence is *uniform* this will be true.

3.2 Uniform Convergence

Supremum and infimum

Definition. Let $A \subset \mathbf{R}$ be a set of real numbers. Let α be the greatest real number so that $x \geq \alpha$ for every $x \in A$. We call α the **infimum** of A. Let β be the smallest real number so that $x \leq \beta$ for every $x \in A$. We call β the **supremum** of A.

Sometimes the infimum and supremum are called the greatest lower bound and least upper bound instead. Note also that these numbers always exist; see the end of the analysis book (the supremum axiom).

Observe the difference between max/min and sup/inf.

Why is minimum and maximum not enough? Well, consider for example the set $A = [0, 1[$. We see that $\min(A) = 0$ and that this is obviously also the infimum of A. However, there is no maximum element in A. The supremum is equal to the value we would need the maximum to attain, that is $\sup(A) = 1$.

Note though, that if there is a maximum element in A, this will also be the supremum. Similarly, if there is a smallest element in A, this will be the infimum.
So with this in mind, consider the linear space of all functions $f : [a, b] \to \mathbf{C}$. We define a normed space $L^\infty[a, b]$ consisting of those functions which has a finite supremum-norm:

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| < \infty.$$

Note that the expression in the left-hand side always exist. Note also that $|f(x)| \leq \|f\|_\infty$ for every $x \in [a, b]$. If we were to restrict our attention to continuous functions on $[a, b]$, we could exchange the supremum for maximum since we know that the maximum for a continuous function on a closed interval is attained (see TATA41).

Uniform convergence

Definition. We say that $u_k \to u$ uniformly on $[a, b]$ as $k \to \infty$ if

$$\lim_{k\to\infty} \|u_k - u\|_\infty = 0.$$

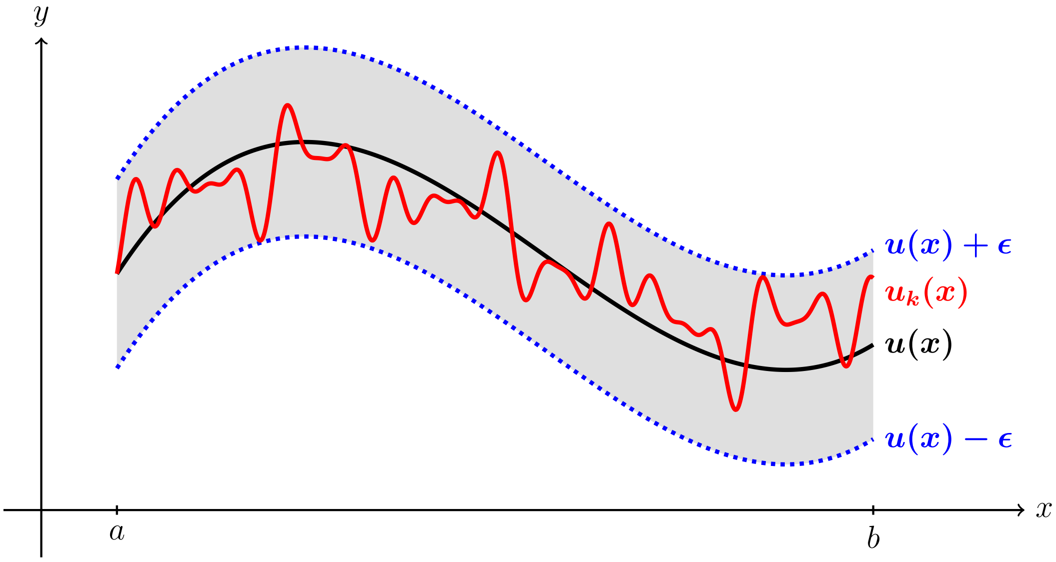
Notice that if $u_k \to u$ uniformly on $[a, b]$, then $u_k \to u$ pointwise on $[a, b]$. The converse, however, does not hold. Let's look at the previous example where $u_k = x^k$ for $0 \leq x \leq 1$. Clearly $u_k(x) \to u(x)$ as $k \to \infty$, where $u(x) = 0$ if $0 \leq x < 1$ and $u(1) = 1$. However, the convergence is *not* uniform:

$$\|u_k - u\|_\infty = \sup_{0\leq x<1} x^k = 1, \quad k = 1, 2, 3, \ldots,$$

so it is not the case that $\|u_k - u\|_\infty$ tends to zero. Therefore the convergence is not uniform. There is another way to see this as well, we'll get to that in the next section when discussing continuity.
By definition, if $u_k \to u$ uniformly on $[a, b]$, this means that for every $\epsilon > 0$, there is some integer N such that

$$k \geq N \quad \Rightarrow \quad \|u_k - u\|_\infty = \sup_{x \in [a,b]} |u_k(x) - u(x)| < \epsilon.$$

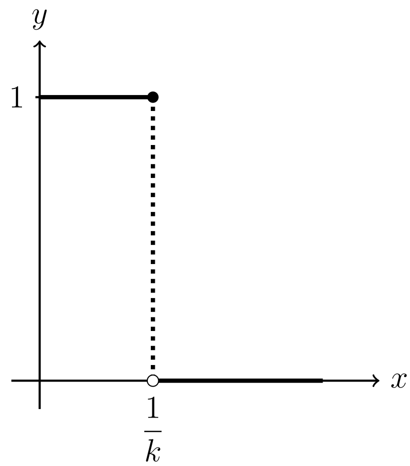
This means that for every $k \geq N$, the difference between $u_k(x)$ and $u(x)$ is less than ϵ *for every* $x \in [a, b]$.



Example

Let $u_k(x) = 0$ if $1/k \leq x \leq 1$ and let $u_k = 1$ if $0 \leq x < 1/k$. Show that $u_k \to u$ pointwise but not uniformly, where $u(x) = 0$ if $x > 0$ and $u(0) = 1$.

Solution. We see that the graph of u_k looks like the figure below.



For any $x \in]0, 1]$, it is clear that $u_k(x) = 0$ if $k > 1/x$. So $u_k(x) \to 0$ for any $x \in]0, 1]$. For $x = 0$ however, there's no $k > 0$ such that $u_k(0) = 0$. The limiting function is $u(x) = 0$ for $x > 0$ and $u(0) = 1$. Hence the convergence cannot be uniform, similar to the previous example.

Example

Show that $u_k(x) = x + \frac{1}{k}x^2$ converges uniformly on $[0, 2]$.

Solution. Clearly $u_k(x) \to x$ as $k \to \infty$ for $x \in [0, 2]$ (for $x \in \mathbf{R}$ really). Hence the pointwise limit is given by $u(x) = x$. Now, observe that

$$|u_k(x) - u(x)| = \left|\frac{1}{k}x^2\right| \leq \frac{1}{k}x^2,$$

so

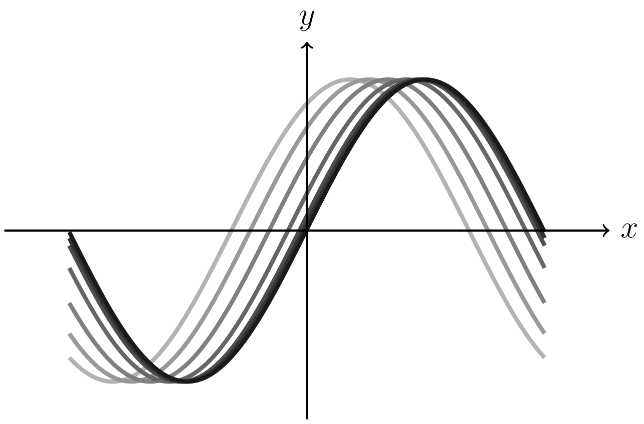
$$\|u_k - u\|_{L^\infty(0,2)} \leq \frac{1}{k}2^2 = \frac{4}{k} \to 0,$$

as $k \to \infty$. Hence the convergence is indeed uniform on $[0, 2]$.

Example

Let $u_k(x) = \sin(x + 1/k)$ for $-\pi \leq x \leq \pi$ and $k > 0$. Does u_k converge uniformly?

Solution. Since sin is continuous, we have $u_k(x) \to \sin x$ for $x \in \mathbf{R}$.



Since sin is differentiable, the mean value theorem implies that

$$\sin(x + 1/k) - \sin x = (x + 1/k - x)\cos\xi,$$

for some ξ between x and $x + 1/k$. Hence

$$|\sin(x + 1/k) - \sin x| \leq |x + 1/k - x| = \frac{1}{k}$$

since $|\cos\xi| \leq 1$. From this it follows that

$$\sup_x |\sin(x + 1/k) - \sin x| \leq \frac{1}{k} \to 0,$$

so the convergence is uniform.

3.3 Continuity and Differentiability

Knowing that a sequence u_k converges pointwise to some function u is not enough to infer that properties like continuity and differentiability are inherited. However, uniform convergence implies that certain properties are inherited by the limiting function.

Theorem. If $u_1, u_2, u_3, \ldots$ is a sequence of continuous functions $u_k : [a, b] \to \mathbf{C}$ and $u_k \to u$ uniformly on $[a, b]$, then u is continuous on $[a, b]$.

Proof. To prove that the limiting function u is continuous, we'll need the δ-ϵ stuff. Let x and x_0 belong to $[a, b]$ and let $\epsilon > 0$. We will show that there exists a $\delta > 0$ so that $|u(x) - u(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$, which proves that u is continuous at x_0. Since x_0 is arbitrary, this proves that u is continuous on $[a, b]$.
Now let's do some triangle inequality magic:

$$\begin{aligned} |u(x) - u(x_0)| &= |u(x) - u_k(x) + u_k(x) - u_k(x_0) + u_k(x_0) - u(x_0)| \\ &\leq |u(x) - u_k(x)| + |u_k(x) - u_k(x_0)| + |u_k(x_0) - u(x_0)| \\ &\leq \|u - u_k\|_\infty + |u_k(x) - u_k(x_0)| + \|u_k - u\|_\infty = 2\|u - u_k\|_\infty + |u_k(x) - u_k(x_0)|, \end{aligned}$$

since $|f(x)| \leq \|f\|_\infty$ for any $f : [a, b] \to \mathbf{C}$. Since $u_k \to u$ uniformly on $[a, b]$, we know that $\|u_k - u\|_\infty \to 0$, so there exists $N \in \mathbf{N}$ so that $\|u_k - u\|_\infty < \epsilon/3$ for $k \geq N$. Furthermore, since u_k is continuous, there exists a $\delta > 0$ so that $|u_k(x) - u_k(x_0)| < \epsilon/3$ whenever $|x - x_0| < \delta$. Thus we obtain that

$$|u(x) - u(x_0)| < 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $|x - x_0| < \delta$. □

Use discontinuity to prove that convergence is *not* uniform

We can exploit the negation of this theorem to prove that a sequence is *not* uniformly convergent. Suppose that

(i) $u_1, u_2, u_3, \ldots$ is a sequence of continuous functions.

(ii) $u_k(x) \to u(x)$ pointwise on $[a, b]$.

(iii) There is some $x_0 \in [a, b]$ where the limiting function u is *not* continuous.

Then the convergence of the sequence can *not* be uniform!

Let's consider the example from the first section again.

Example

Let $u_k(x) = x^k$ if $0 \leq x \leq 1$, $k = 1, 2, 3, \ldots$. Then $u_k(x) \to 0$ for $0 \leq x < 1$ and $u_k(x) \to 1$ when $x = 1$. Clearly u_k is continuous on $[0, 1]$ for every k, but the *limiting* function is discontinuous at $x = 1$. Hence the convergence *cannot* be uniform!

It's not just the continuity that's easier to infer, we can also work with integrals like they were sums and exchange the order of integration and taking limits.

Theorem. Suppose that $u_1, u_2, u_3, \ldots$ is a sequence of continuous functions $u_k : [a, b] \to \mathbf{C}$ and that $u_k \to u$ uniformly on $[a, b]$. Then

$$\lim_{k\to\infty} \int_a^b u_k(x)\,dx = \int_a^b \lim_{k\to\infty} u_k(x)\,dx = \int_a^b u(x)\,dx.$$

Proof. Assume that $b > a$. Since the integral is monotonous (we get a bigger value when moving the modulus inside), we see that

$$\begin{aligned}\left|\int_a^b u_k(x)\,dx - \int_a^b u(x)\,dx\right| &= \left|\int_a^b \big(u_k(x) - u(x)\big)\,dx\right| \le \int_a^b \big|u_k(x) - u(x)\big|\,dx \\ &\le \int_a^b \big\|u_k - u\big\|_\infty\,dx = \big\|u_k - u\big\|_\infty \int_a^b dx \\ &= \big\|u_k - u\big\|_\infty (b-a) \to 0, \text{ as } k \to \infty,\end{aligned}$$

since $\|u_k - u\|_\infty$ is independent of x.

Remark. There are other results of this type with much weaker assumptions. Continuity is not necessary (it is enough the it is a sequence of *integrable* functions) and the uniform convergence can be exchanged for weaker types of convergence as well (*dominated convergence*).

Example

Find the value of $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{nx+1}{nx^2+x+n}\,dx$.

Solution. Let $u_n(x) = \dfrac{nx+1}{nx^2+x+n}$, $n = 1, 2, 3, \ldots$ and $0 \le x \le 1$. Then

$$\frac{nx+1}{nx^2+x+n} = \frac{x+1/n}{x^2+1+x/n} \to \frac{x}{x^2+1},$$

as $n \to \infty$. Moreover,

$$\begin{aligned}\left|\frac{nx+1}{nx^2+x+n} - \frac{x}{x^2+1}\right| &= \left|\frac{(nx+1)(x^2+1) - x(nx^2+x+n)}{(x^2+1)(nx^2+x+n)}\right| \\ &= \left|\frac{1}{(x^2+1)(nx^2+x+n)}\right| = \frac{1}{n}\left|\frac{1}{(x^2+1)(x^2+x/n+1)}\right| \le \frac{1}{n}\end{aligned}$$

since $1 + x^2 \ge 1$ and $x^2 + x/n + 1 \ge 1$. Clearly this means that

$$\sup_{0\le x\le 1}\left|\frac{nx+1}{nx^2+x+n} - \frac{x}{x^2+1}\right| \le \frac{1}{n} \to 0,$$

as $n \to \infty$. The convergence is therefore uniform and

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \frac{nx+1}{nx^2+x+n}\,dx &= \int_0^1 \lim_{n\to\infty}\frac{nx+1}{nx^2+x+n}\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \\ &= \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{\ln 2}{2}.\end{aligned}$$

Integrals and uniform limits

Notice the steps in the previous example:

(i) Find the pointwise limit $u(x)$ of $u_k(x)$.

(ii) Find a *uniform* bound for $|u_k(x) - u(x)|$ that tends to zero as $k \to \infty$ (independently of x).

(iii) Deduce that $u_k \to u$ uniformly.

(iv) Move the limit inside the integral, effectively replacing $\lim u_k$ by u, and calculate the resulting integral.

There are no short-cuts. Without a clear motivation about the fact that we have uniform convergence and what this means, the result will be zero points (even with the "right answer").

So what about taking derivatives? That's slightly more difficult.

Theorem. Let $u_1, u_2, u_3, \ldots$ be a sequence of differentiable functions $u_k : [a, b] \to \mathbf{C}$. If $u_k \to u$ pointwise on $[a, b]$ and $u_k' \to v$ uniformly on $[a, b]$, where v is continuous, then u is differentiable on $[a, b]$ and $u' = v$.

Proof. Since u_k is differentiable, it is clear that

$$u_k(x) - u_k(a) = \int_a^x u_k'(t)\, dt, \quad x \in [a, b].$$

By assumption, $u_k' \to v$ uniformly on $[a, b]$, so the previous theorem implies that

$$\int_a^x u_k'(t)\, dt \to \int_a^x v(t)\, dt.$$

Since $u_k \to u$ pointwise on $[a, b]$, we must have that $u(x) - u(a) = \displaystyle\int_a^x v(t)\, dt$. We know that v is continuous, so the fundamental theorem of calculus proves that $u' = v$ on $[a, b]$. □

3.4 Series

Let $u_0, u_1, u_2, u_3, \ldots$ be a sequence of functions $u_k : I \to \mathbf{C}$, where I is some set. As stated earlier, we define the series $S(x) = \displaystyle\sum_{k=0}^{\infty} u_k(x)$ for those x where the limit exist. This is the pointwise limit of the partial sums $S_n(x) = \displaystyle\sum_{k=0}^{n} u_k(x)$. When does the sequence $S_0, S_1, S_2, \ldots$ converge uniformly? And why would we be interested in this? Well, a rather typical question is if the series converge to something continuous, or differentiable. And whether we can take the derivative of a series — or an integral — *termwise*. In other words, when does a series behave like we are used to when working with a power series? Uniform convergence is a tool to obtain many of these properties and one way of proving uniform convergence is the Weierstrass M-test.

Weierstrass M-test

Theorem. Let $I \subset \mathbf{R}$. Suppose that there exists positive constants M_k, $k = 1, 2, \ldots$, such that $|u_k(x)| \le M_k$ for $x \in I$. If $\sum_{k=1}^{\infty} M_k < \infty$, then $\sum_{k=1}^{\infty} u_k(x)$ converges uniformly on I.

Proof. Since $|u_k(x)| \le M_k$ and $\sum_{k=1}^{\infty} M_k$ is convergent, it is clear that

$$u(x) = \sum_{k=1}^{\infty} u_k(x)$$

exists for every $x \in I$. Now

$$\begin{aligned}\left\| u(x) - \sum_{k=1}^{n} u_k(x) \right\|_{\infty} &= \left\| \sum_{k=1}^{\infty} u_k(x) - \sum_{k=1}^{n} u_k(x) \right\|_{\infty} = \left\| \sum_{k=n+1}^{\infty} u_k(x) \right\|_{\infty} \\ &\le \sum_{k=n+1}^{\infty} \| u_k(x) \|_{\infty} \le \sum_{k=n+1}^{\infty} M_k \to 0,\end{aligned}$$

as $k \to \infty$. By definition, this implies that the series is uniformly convergent. □

By considering the sequence of partial sums $S_n(x)$, $n = 0, 1, 2, \ldots$, of a uniformly convergent series $\sum_{k=0}^{\infty} u_k(x)$, we can express some of the results from the preceding sections in a more convenient form for working with function series.

Series and uniform convergence

Suppose that $u(x) = \sum_{k=0}^{\infty} u_k(x)$ is uniformly convergent for $x \in [a, b]$. If $u_0, u_1, u_2, \ldots$ are continuous functions on $[a, b]$, then the following holds.

(i) The series u is a continuous function on $[a, b]$.

(ii) We can exchange the order of summation and integration:

$$\int_c^d u(x)\, dx = \int_c^d \left(\sum_{k=0}^{\infty} u_k(x) \right) dx = \sum_{k=0}^{\infty} \int_c^d u_k(x)\, dx, \quad \text{for } a \le c < d \le b.$$

(iii) If in addition $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then

$$u'(x) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} u_k(x)\, dx \right) = \sum_{k=0}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=0}^{\infty} u_k'(x), \quad x \in [a, b].$$

Note that all of the above also holds for series of the form $\sum_{k=-\infty}^{\infty} u_k(x)$ when using symmetric partial sums $S_n(x) = \sum_{k=-n}^{n} u_k(x)$.

Example

Let $0 < a < 1$ and $ab > 1$. Show that $u(x) = \sum_{k=1}^{\infty} a^k \sin(b^k \pi x)$ is continuous.

Solution. We see that

$$\left|a^k \sin(b^k \pi x)\right| \le a^k, \quad k = 1, 2, 3, \dots,$$

since $|\sin(b^k \pi x)| \le 1$. Since $\sum_{k=1}^{\infty} a^k$ is a geometric series with quotient a and $|a| < 1$, we know that this series is convergent. Thus, by Weierstrass' M-test, it follows that the original series is convergent (absolutely) and that u is continuous.

Note that we didn't calculate the exact $\|\cdot\|_\infty$ norm (well we actually did but we never claimed that the bound was the actual maximum). We just estimated with something that is an upper bound. This is typical (and usually enough). This series is especially interesting since it is an example of a function that is continuous, but *nowhere* differentiable (it is usually referred to as Weierstrass' function). The fact that it is not differentiable is not obvious, but it shows that uniform convergence *isn't* enough to ensure that the limit of something differentiable is differentiable.

In fact, the Weierstrass function does not even have one-sided derivatives (finitely) at any point. So this is an example of a continuous function that definitely does not belong to E'.

3.5 The Dirichlet Kernel

Consider the complex Fourier series of u. Let us write out and exchange the order of summation and integration according to

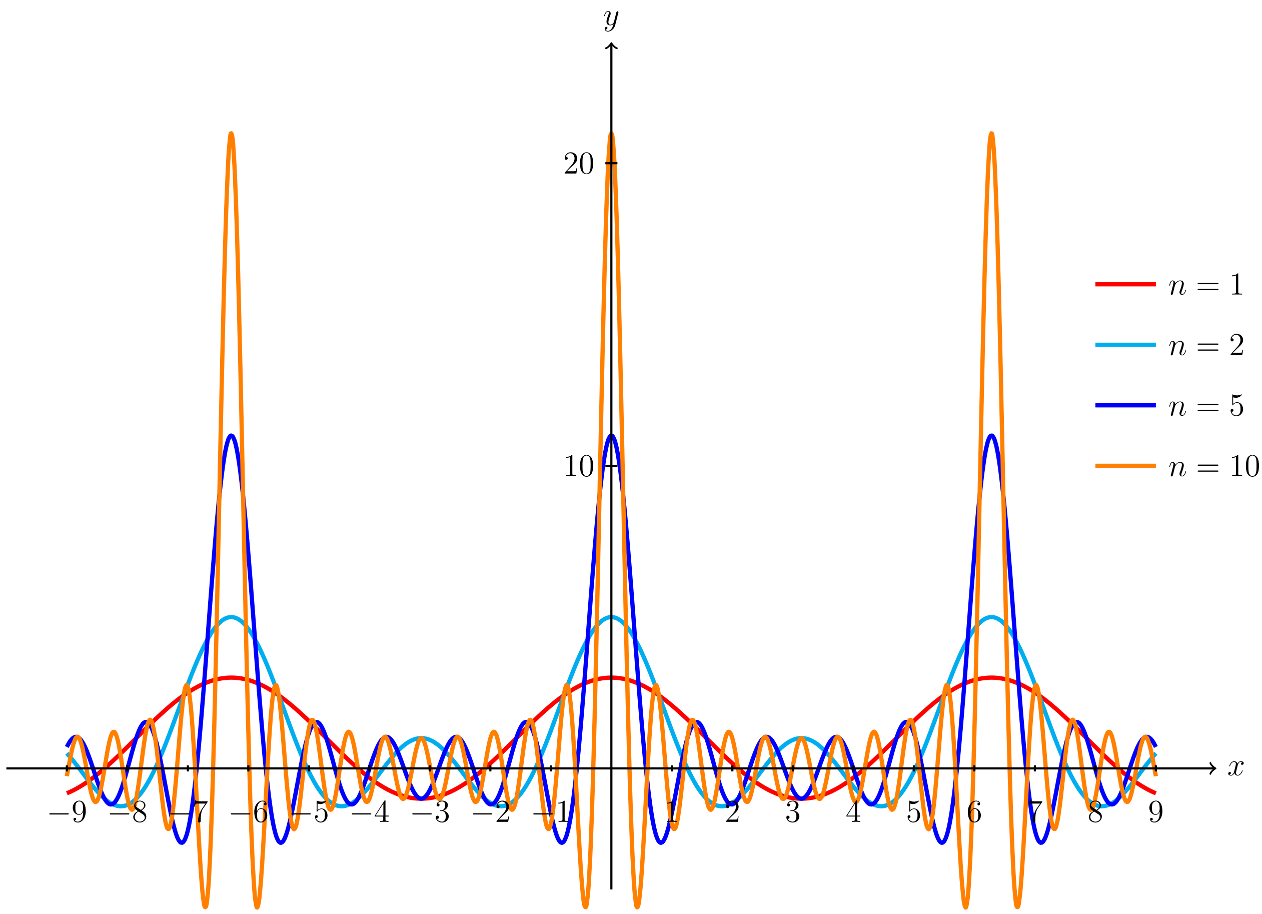
$$\begin{aligned} S_n(x) &= \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_T u(t) e^{-ikt}\, dt \right) e^{ikx} = \frac{1}{2\pi} \int_T u(t) \left(\sum_{k=-n}^{n} e^{-ikt} e^{ikx} \right) dt \\ &= \frac{1}{2\pi} \int_T u(t) \left(\sum_{k=-n}^{n} e^{ik(x-t)} \right) dt. \end{aligned}$$

The sum in the parentheses is usually referred to as the Dirichlet kernel.

The Dirichlet kernel

Definition. We define the **Dirichlet kernel** by

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}, \ x \in \mathbf{R}, \ n = 1, 2, 3, \dots$$



This means that we can write

$$S_n u(x) = \frac{1}{2\pi}\int_T u(t)D_n(x-t)\,dt = \frac{1}{2\pi}\int_T u(s+x)D_n(-s)\,ds = \frac{1}{2\pi}\int_T u(s+x)D_n(s)\,ds,$$

so the partial sums of the Fourier series is given by a *convolution* of u with the Dirichlet kernel (we will get back to convolutions later on). In the first equality, we changed variables ($t-x=s$) and used the fact that u and D_n are periodic so that we can use the same domain of integration and also that D_n is an even function. The reason for this representation of the partial sums will become clear below.
Let us collect some properties of the Dirichlet kernel.

Theorem.

(i) $D_n(2k\pi) = 2n+1$, $k \in \mathbf{Z}$.

(ii) $D_n(x) = \dfrac{\sin((2n+1)x/2)}{\sin(x/2)}$, $x \neq 2k\pi$, $k \in \mathbf{Z}$.

(iii) $\displaystyle\int_T D_n(x)\,dx = 2\pi$.

Proof.

(i) Since $e^{i2k\pi} = 1$ for $k \in \mathbf{Z}$, it is clear that $D_n(2k\pi) = 2n+1$ since there are $2n+1$ terms in the sum $D_n(x)$.

(ii) For $x \neq 2k\pi$, we observe that $D_n(x)$ is a geometric sum with quotient $e^{ix} \neq 1$, first term e^{-inx} and $2n+1$ terms, so

$$D_n(x) = e^{-inx} \cdot \frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} = e^{-i(n+1/2)x} \cdot \frac{e^{i(2n+1)x} - 1}{e^{ix/2} - e^{-ix/2}} = \frac{e^{i(n+1/2)x} - e^{-i(n+1/2)x}}{e^{ix/2} - e^{-ix/2}}$$
$$= \frac{\sin((n+1/2)x)}{\sin(x/2)},$$

which is the same expression as given in the statement above.

(iii) We see that

$$\int_{-\pi}^{\pi} D_n(x)\,dx = \int_{-\pi}^{\pi} \left(\sum_{k=-n}^{n} e^{ikx} \right) dx = \int_{-\pi}^{\pi} \left(1 + \sum_{k=1}^{n} \left(e^{ikx} + e^{-ikx}\right) \right) dx$$
$$= 2\pi + \sum_{k=1}^{n} 2 \int_{-\pi}^{\pi} \cos kx\,dx = 2\pi,$$

since all the integrals in the sum are equal to zero. □

3.6 Pointwise Convergence

We now have the tools to prove that for a function in the space E' (so left- and righthand derivatives exist), the Fourier series actually converges to something that involves the function.

Pointwise convergence (Dirichlet's theorem)

Theorem. Let $u \in E'$. Then

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{u(x^+) + u(x^-)}{2}, \quad x \in [-\pi, \pi].$$

In other words, the Fourier series of u converges pointwise to $\dfrac{u(x^+) + u(x^-)}{2}$ for $x \in [-\pi, \pi]$. In particular, if u also is continuous at x, then $\lim_{n\to\infty} S_n(x) = u(x)$.

Notice the following.

(i) It is sufficient for $u \in E$ (not E') to have left- and righthand derivatives at a specific point x for

$$\lim_{n\to\infty} S_n(x) = \frac{u(x^+) + u(x^-)}{2}$$

to hold at the point x. The condition that $u \in E'$ ensures that this is true for all x.

(ii) The number $(u(\pi^+)+u(\pi^-))/2$ is defined since u is 2π-periodic so that $u(\pi^+) = u((-\pi)^+)$ (the righthand limit at π must be equal to the righthand limit at $-\pi$) and similarly for $u((-\pi)^-)$.

Proof. Let $x \in [-\pi, \pi]$ be fixed (meaning that we won't change the value). We will prove that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^\pi u(x+t) D_n(t)\, dt = \frac{u(x^+)}{2}. \tag{3.1}$$

A completely analogous argument would show that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^0 u(x+t) D_n(t)\, dt = \frac{u(x^-)}{2}$$

and these two limits taken together proves the statement in the theorem.
First we note that

$$\frac{1}{2\pi} \int_0^\pi u(x+t) D_n(t)\, dt - \frac{u(x^+)}{2} = \frac{1}{2\pi} \int_0^\pi \big(u(x+t) - u(x^+)\big) D_n(t)\, dt$$

since $D_n(t)$ is an even function so $\dfrac{1}{2\pi} \displaystyle\int_0^\pi D_n(t)\, dt = \dfrac{1}{2}$ (see the theorem about the Dirichlet kernel above). The same theorem also provides the identity $D_n(x) = \dfrac{\sin((2n+1)x/2)}{\sin(x/2)}$, so

$$\begin{aligned}\big(u(x+t) - u(x^+)\big) D_n(t) &= \big(u(x+t) - u(x^+)\big) \frac{\sin((2n+1)t/2)}{\sin(t/2)} \\ &= \frac{u(x+t) - u(x^+)}{t} \cdot \frac{t}{\sin(t/2)} \cdot \sin\left(nt + t/2\right).\end{aligned}$$

Since $u \in E'$, we know that the righthand derivative of u at x exists, so

$$\frac{u(x+t) - u(x^+)}{t} \cdot \frac{t}{\sin(t/2)} \to 2D^+u(x). \tag{3.2}$$

This means that the expression in the left-hand side of (3.2) is bounded on $[0, \pi]$ (since it is quite nice outside of the origin). Hence it also belongs to $L^2(0, \pi)$ and E since u is piecewise continuous. Letting

$$v(t) = \begin{cases} \dfrac{u(x+t) - u(x^+)}{t} \cdot \dfrac{t}{\sin(t/2)}, & 0 \le t \le \pi, \\ 0, & -\pi < t < 0, \end{cases}$$

it is clear that $v \in E \subset L^2(-\pi, \pi)$. By the Riemann-Lebesgue lemma, it now follows that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^\pi \big(u(x+t) - u(x^+)\big) D_n(t)\, dt = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^\pi v(t) \sin\left((n+1/2)t\right)\, dt = 0,$$

which proves that (3.1) holds. □

Example

Find the Fourier series for the sign-function $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = 1$ if $x > 0$. Prove when and to what it converges to.

Solution. The Fourier coefficients can be calculated as follows:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x)\, dx = 0$$

since sgn is odd, and for $k \neq 0$,

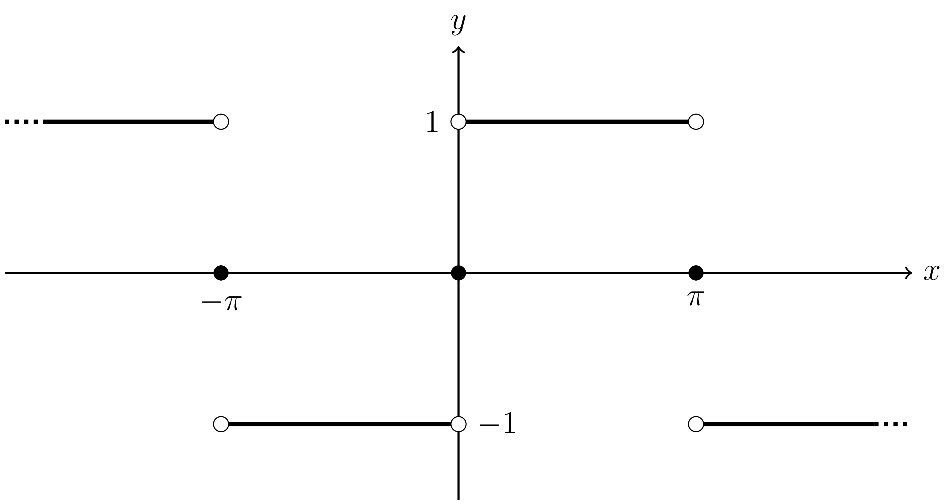
$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) e^{-ikx}\, dx = \frac{1}{2\pi} \left(\int_{-\pi}^{0} -e^{-ikx}\, dx + \int_{0}^{\pi} e^{-ikx}\, dx \right) \\ &= \frac{1}{2\pi} \left(\left[\frac{e^{-ikx}}{ik} \right]_{-\pi}^{0} + \left[-\frac{e^{-ikx}}{ik} \right]_{0}^{\pi} \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{ik} - \frac{(-1)^k}{ik} - \frac{(-1)^k}{ik} + \frac{1}{ik} \right) = \frac{2(1-(-1)^k)}{2\pi \cdot ik} \\ &= i\, \frac{(-1)^k - 1}{\pi k}. \end{aligned}$$

Hence

$$u(x) \sim \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} i\, \frac{(-1)^k - 1}{\pi k} e^{ikx}.$$

For $-\pi < x < 0$ and $0 < x < \pi$, u is continuously differentiable so the Fourier series converges to $u(x)$. For $x = 0$, both right- and lefthand derivative exists (both are zero) so the Fourier series converges to $(u(0^-) + u(0^+))/2 = (-1+1)/2 = 0$. This happens to be equal to $\operatorname{sgn}(0)$, but this is more of a coincidence. Indeed, we could redefine $\operatorname{sgn}(0) = A$ for any number A we would like and the Fourier series would still converge to 0. For the endpoints, the right- and lefthand derivatives exist (respectively) so the Fourier series converges to $(u(-\pi)+u(\pi))/2 = 0$. Note the analogous situation as that which occurs at $x = 0$.

We can now draw the Fourier series since we have analyzed in detail what the series converges to at every point.



Example

Recall that if $u(x) = x^2$ for $-\pi < x < \pi$, then $u(x) \sim \dfrac{\pi^2}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \dfrac{2(-1)^k}{k^2} e^{ikx}$. Use this to evaluate the series $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^2}$.

Solution. Since x^2 is continuously differentiable on $]-\pi,\pi[$, continuous on $[-\pi,\pi]$ with right- and lefthand derivatives at the endpoints (respectively), and $(-\pi)^2 = \pi^2$, it is clear that the Fourier series of $u(x)$ converges to $u(x)$ for any $x \in [-\pi,\pi]$. Especially this holds for $x = 0$. Therefore

$$0 = u(0) = \frac{\pi^2}{3} + \sum_{k=-\infty,\, k\neq 0}^{\infty} \frac{2(-1)^k}{k^2} e^{ik\cdot 0} = \frac{\pi^2}{3} + 2\sum_{k=1}^{\infty} \frac{2(-1)^k}{k^2} = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2},$$

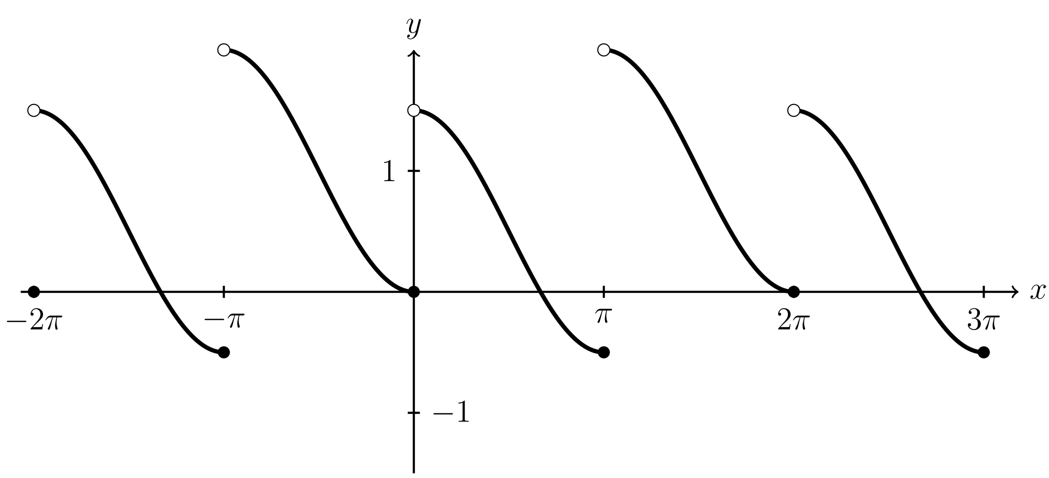
so $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^2} = -\dfrac{\pi^2}{12}$.

Could you use the stuff from the previous example to calculate $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$?

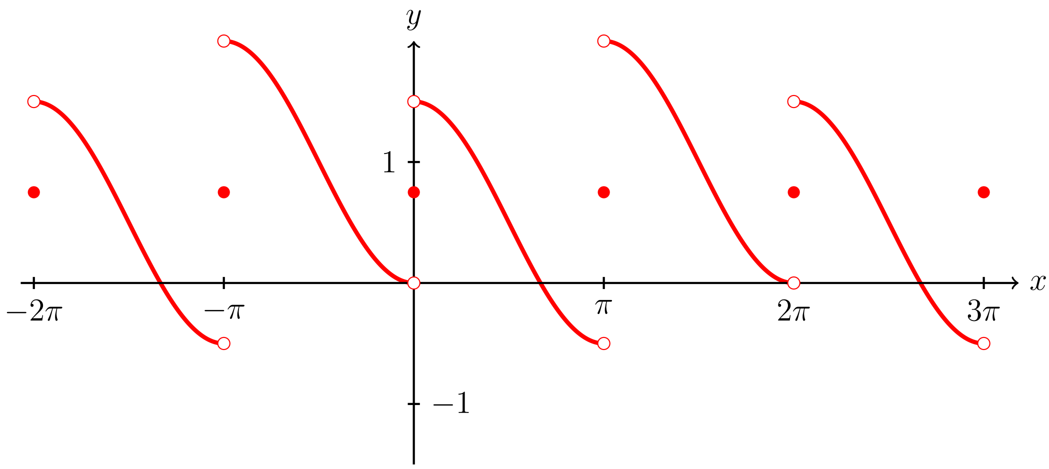
Example

Let $u(x) = \dfrac{1}{2} + \cos x$ for $0 < x \leq \pi$, $u(0) = 0$, and $u(x) = 1 - \cos x$ for $-\pi < x < 0$. Show that the Fourier series for the periodic extension of u converges and find the limit of the Fourier series. Where is it equal to u? What's the value of the Fourier series at π? At 2π? Can the convergence be uniform?

Solution. Note that the we do *not* need to find the Fourier series to answer this question. The function is piecewise continuous and has right- and left-hand derivatives at all points. Hence, by Dirichlet's theorem above, we know that the Fourier series converges to $(u(x^+) + u(x^-))/2$ at all points. So the Fourier series exists (since $u \in E$) and is convergent. As to what the limit $S(x)$ of the Fourier series actually is, let's first draw a graph of the function u (being very careful at the exception points).



So, again, we know that $S(x) = \dfrac{u(x^+) + u(x^-)}{2}$ at every $x \in \mathbf{R}$ since $u \in E$ has right- and lefthand derivatives at every point (is this clear?). For points of continuity of u, that means that $S(x) = u(x)$. For the "jump" points, we take the mean value. This produces the graph below.



What you shouldn't miss here is the fact that it's completely irrelevant what value the function u takes at a single point. It's only the limits of the function that has any effect on the limit of the Fourier series.
From this graph we immediately find that

$$S(\pi) = \frac{u(\pi^-) + u(\pi^+)}{2} = \frac{-1/2 + 2}{2} = \frac{3}{4}$$

and that

$$S(2\pi) = \frac{u(2\pi^-) + u(2\pi^+)}{2} = \frac{0 + 3/2}{2} = \frac{3}{4}.$$

Note in particular that we get the same value, but for two completely different reasons. This is a coincidence (well.. the reason is the symmetry of the function u).
The convergence of the partial sums $S_n(x)$ can not be uniform on any set that includes a point $x = k\pi$ for some integer k. The reason for this is that $S(x)$ is discontinuous at such points, whereas the partial sums $S_n(x)$ (being trigonometric polynomials) are continuous functions on $\mathbf{R}$. Having the limiting function $S(x)$ being discontinuous would violate the convergence being uniform.

Chapter 4

Stronger Types of Convergence

"I did nothing! The pavement was his enemy."
—Julius Benedict

4.1 Absolute Convergence

So we have obtained sufficient conditions for the Fourier series to converge to the mean value of the left- and righthand limits of the function. Note though that we have said nothing about necessary conditions (and this is not really something we will be able to cover in this course). So let's look in the other direction instead: when can we obtain a stronger type of convergence? Suppose that $\{c_k\}_{k\in\mathbf{Z}} \in l^1(\mathbf{Z})$, meaning that the series $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, which implies that the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges absolutely. This implies that we have uniform convergence. Let's formulate a theorem.

Theorem. Suppose that $\sum_{k=-\infty}^{\infty} |c_k| < \infty$. Then $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly.

Proof. Note that

$$S(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

converges for every $x \in \mathbf{R}$ since

$$|S(x) - S_n(x)| = \left| \sum_{|k|>n} c_k e^{ikx} \right| \leq \sum_{|k|>n} |c_k| \to 0,$$

as $n \to \infty$. Note also that the last series is independent of x, which implies uniform convergence:

$$\sup_x |S(x) - S_n(x)| \leq \sum_{|k|>n} |c_k| \to 0,$$

as $n \to \infty$. □

4.2 A Case Study: $u(x) = x$

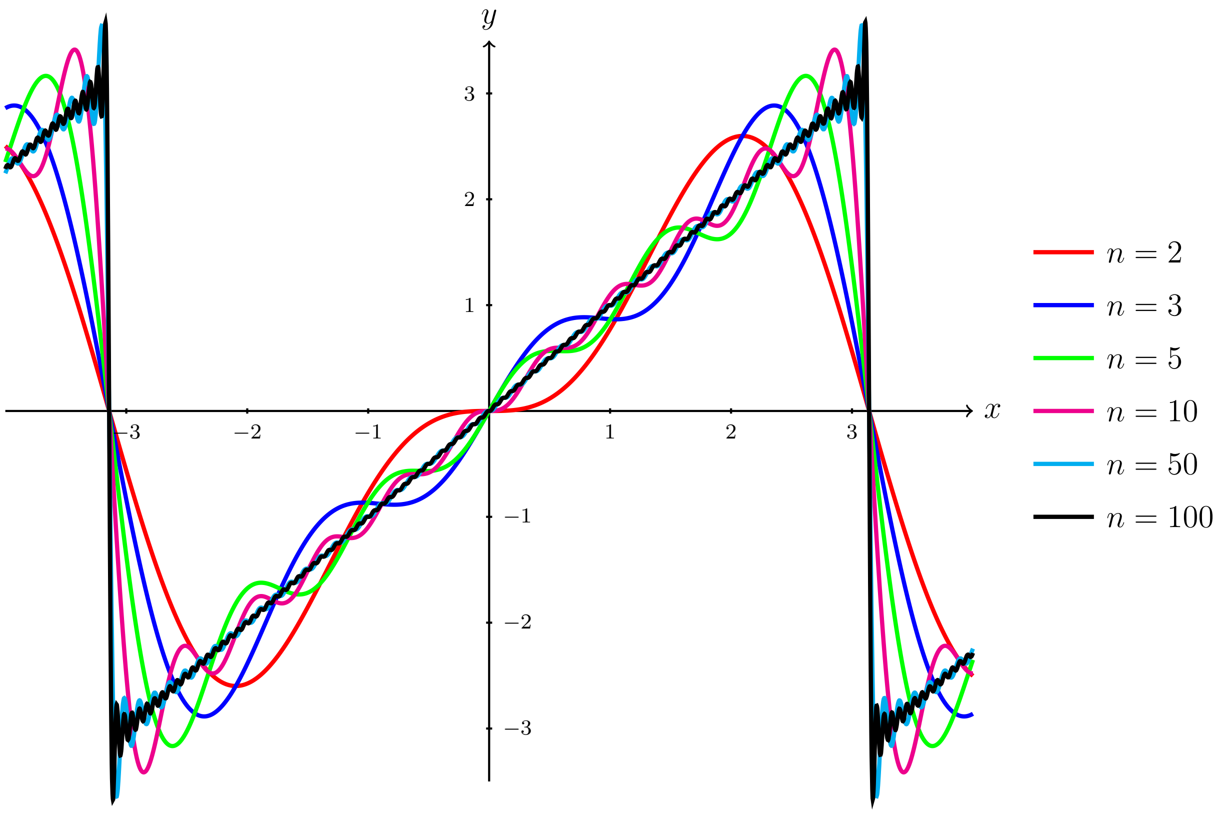
Consider the case when $u(x) = x$ for $-\pi < x < \pi$ (and periodically extended to $\mathbf{R}$). You've seen this before, but let's find the Fourier coefficients. Clearly $c_0 = 0$ (the function is odd) and for $k \neq 0$, integration by parts yields

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-ikx}\,dx = \frac{1}{2\pi}\left(\left[\frac{xe^{-ikx}}{-ik}\right]_{-\pi}^{\pi} + \frac{1}{ik}\int_{-\pi}^{\pi} e^{-ikx}\,dx\right) = \frac{1}{2\pi}\left(\frac{2\pi(-1)^k}{-ik} + 0\right)$$
$$= \frac{(-1)^{k+1}}{ik}.$$

Since u is continuously differentiable for $-\pi < x < \pi$, we know from the previous lecture that

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = x, \quad -\pi < x < \pi.$$

We also know that the Fourier series converges to 0 at $x = \pm\pi$. Since the limit function is discontinuous at $\pm\pi$, the convergence can *not* be uniform on $[-\pi, \pi]$, but there's a possibility that the convergence *is* uniform for $a < x < b$ with $-\pi < a < b < \pi$.



Let's show the following fact.

Theorem. For $0 < b < \pi$, the Fourier series for $u(x) = x$, $-\pi < x < \pi$, converges uniformly on $[-b, b]$.

Proof. Suppose that $m, n \in \mathbf{N}$ and that $m > n$. Let

$$S_l(x) = \sum_{k=-l}^{l} c_k e^{ikx}, \quad l = 1, 2, 3, \ldots,$$

be the partial sums for the Fourier series of $u(x) = x$, $-\pi < x < \pi$. We will show that

$$\sup_{x \in [-b,b]} |S_m(x) - S_n(x)| \to 0,$$

as $m, n \to \infty$. This is sufficient for obtaining uniform convergence (and is known as Cauchy's criterion for uniform convergence). First we note that

$$\begin{aligned} S_m(x) - S_n(x) &= \sum_{k=-m}^{m} c_k e^{ikx} - \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-m}^{-(n+1)} c_k e^{ikx} - \sum_{k=n+1}^{m} c_k e^{ikx} \\ &= \sum_{k=n+1}^{m} \left(\frac{(-1)^{k+1}}{ik} e^{ikx} + \frac{(-1)^{-k+1}}{-ik} e^{-ikx} \right) = \sum_{k=n+1}^{m} \frac{(-1)^{k+1}}{ik} \left(e^{ikx} - e^{-ikx}\right). \end{aligned}$$

We need to exploit the fact that the terms are both positive and negative to show that this is small. For this purpose, we observe that $\delta = \cos(b/2) > \cos(\pi/2) = 0$. If we were to multiply a term in the series by $\cos(x/2)$, we would obtain

$$\begin{aligned} \cos\left(\frac{x}{2}\right) \frac{(-1)^{k+1}}{ik} \left(e^{ikx} - e^{-ikx}\right) &= \frac{(-1)^{k+1}}{i2k} \left(e^{ix/2} + e^{-ix/2}\right)\left(e^{ikx} - e^{-ikx}\right) \\ &= \frac{(-1)^{k+1}}{i2k} \left(e^{i(k+1/2)x} + e^{i(k-1/2)x} - e^{-i(k-1/2)x} - e^{-i(k+1/2)x}\right) \\ &= \frac{(-1)^{k+1}}{k} \left(\sin\left(k + \frac{1}{2}\right) x + \sin\left(k - \frac{1}{2}\right) x\right). \end{aligned}$$

Examining the sum of these terms more closely, we find that

$$\sum_{k=n+1}^{m} \frac{(-1)^{k+1}}{k} \left(\sin\left(k + \frac{1}{2}\right) x + \sin\left(k - \frac{1}{2}\right) x\right)$$

is equal to

$$\begin{aligned} (-1)^{n+1} \Bigg(&\frac{\sin\left(n + \frac{3}{2}\right) x + \sin\left(n + \frac{1}{2}\right) x}{n+1} - \frac{\sin\left(n + \frac{5}{2}\right) x + \sin\left(n + \frac{3}{2}\right) x}{n+2} \\ &+ \frac{\sin\left(n + \frac{7}{2}\right) x + \sin\left(n + \frac{5}{2}\right) x}{n+3} - \frac{\sin\left(n + \frac{9}{2}\right) x + \sin\left(n + \frac{7}{2}\right) x}{n+4} \\ &+ \cdots \pm \frac{\sin\left(m - \frac{1}{2}\right) x + \sin\left(m - \frac{3}{2}\right) x}{m-1} \mp \frac{\sin\left(m + \frac{1}{2}\right) x + \sin\left(m - \frac{1}{2}\right) x}{m} \Bigg). \end{aligned}$$

We can rearrange the terms in the parenthesis as

$$\frac{\sin\left(n + \frac{1}{2}\right) x}{n+1} + (-1)^{n+1} \sum_{k=n+1}^{m-1} (-1)^k \left(\frac{1}{k} - \frac{1}{k+1}\right) \sin\left(k + \frac{1}{2}\right) x + (-1)^m \frac{\sin\left(m + \frac{1}{2}\right) x}{m}.$$

Using the fact that $|\sin t| \leq 1$ for $t \in \mathbf{R}$, we now obtain that

$$\begin{aligned}|S_m(x) - S_n(x)| &\leq \delta^{-1}\left(\frac{1}{n+1} + \sum_{k=n+1}^{m-1}\left|\frac{1}{k} - \frac{1}{k+1}\right| + \frac{1}{m}\right)\\ &\leq \delta^{-1}\left(\frac{1}{n+1} + \sum_{k=n+1}^{m-1}\frac{1}{k(k+1)} + \frac{1}{m}\right) \leq \quad \delta^{-1}\left(\frac{2}{n} + \sum_{k=n+1}^{\infty}\frac{1}{k^2}\right) \to 0,\end{aligned}$$

as $n \to \infty$ since the series of $1/k^2$ is convergent and $2/n \to 0$. □

So that was awesome (or absolutely positively horrifying). We'll need the Fourier expansion of x later on in this lecture.

4.3 Uniform Convergence

So as seen from the previous case study, proving uniform convergence directly can be rather cumbersome. And obviously, demanding that we have absolute convergence is rather restrictive. We would prefer less draconian requirements that are easier to verify. Too much to ask for? Not really!

Theorem. Suppose that u is continuous on $[-\pi, \pi]$, that $u(-\pi) = u(\pi)$ and that $u' \in E$. Then the Fourier series of u converges uniformly to u on $[-\pi, \pi]$.

Proof. Since $u' \in E$, we know that u' has a Fourier series

$$u'(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx},$$

where

$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(t)e^{-ikt}\,dt, \quad k \in \mathbf{Z}.$$

Furthermore, the fact that $u(-\pi) = u(\pi)$ implies that $d_0 = 0$:

$$d_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(t)\,dt = \frac{u(\pi) - u(-\pi)}{2\pi} = 0.$$

Now, since $u' \in E$, it is clear that $u \in E'$, which implies that u has a convergent Fourier series

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where the equality follows from the fact that u is continuous and

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt, \quad k \in \mathbf{Z}.$$

How are c_k and d_k related? Assuming that $k \neq 0$, we have

$$\begin{aligned}d_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u'(t)e^{-ikt}\,dt = \text{/ I.B.P. /} = \frac{1}{2\pi}\left(\left[u(t)e^{-ikt}\right]_{-\pi}^{\pi} + ik\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt\right)\\ &= ikc_k\end{aligned}$$

since $u(\pi)e^{-ik\pi} = u(-\pi)e^{ik\pi}$ for $k \in \mathbf{Z}$. Now, Bessel's inequality shows that

$$\sum_{k=-\infty}^{\infty} |d_k|^2 \leq \|u'\|^2_{L^2(-\pi,\pi)} < \infty,$$

and since $d_k = ikc_k$, this implies that

$$\sum_{k=-\infty}^{\infty} k^2|c_k|^2 \leq \|u'\|^2_{L^2(-\pi,\pi)} < \infty.$$

Note that we could have used Parseval's identity in the place of Bessel's inequality, but we haven't shown why this holds yet. Now, by the Cauchy-Schwarz inequality,

$$\sum_{k=1}^{\infty} |c_k| = \sum_{k=1}^{\infty} \left| \frac{1}{k} \cdot kc_k \right| \leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{1/2} \left(\sum_{k=1}^{\infty} k^2|c_k|^2 \right)^{1/2} < \infty,$$

and similarly for $k < 0$. By Weierstrass' M-test, it now follows that

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is uniformly convergent since $|c_k e^{ikx}| \leq |c_k|$ due to the fact that $|e^{ikx}| = 1$.

Smoothness and convergence

The proof of the previous theorem exploits the fact that u is quite smooth (meaning that u' exists) to obtain that the Fourier coefficients of u tend to zero faster than if u was not smooth. This is something important when it comes to Fourier analysis: a smoother function provides better convergence. What could we do if u is twice differentiable?

We can also "localize" the previous theorem to show that we actually have uniform convergence on any interval $[a, b] \subset [-\pi, \pi]$ such that u is continuous with a piecewise continuous derivative.

Theorem. Suppose that u is continuous on $[a, b] \subset [-\pi, \pi]$ and that $u, u' \in E[-\pi, \pi]$. Then the Fourier series of u converges uniformly to u on $[a, b]$.

Proof. We will use that fact that the Fourier series for $v(x) = x$ converges uniformly on every interval $[-c, c] \subset]-\pi, \pi[$ to modify $u(x)$ so that we can apply the previous (global) uniform convergence result. To this end, let $v(x) = x$ for $-\pi < x < \pi$ and $v(\pm\pi) = 0$. Moreover, let

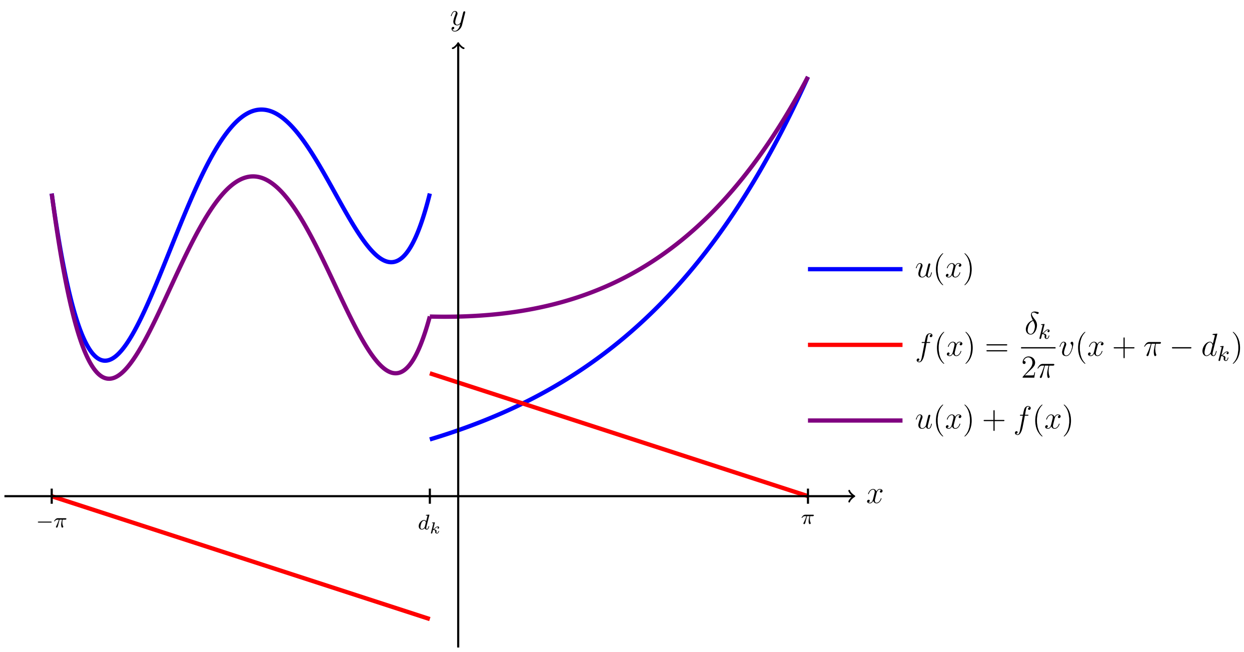
$$-\pi < d_1 < d_2 < \cdots < d_n \leq \pi,$$

where d_k are the points of discontinuity of u (where the function "jumps"). We can also redefine $u(\pm\pi)$ so that these values are equal (and thereby possible adding a new point d_k). This will not affect the result since $[a, b] \subset]-\pi, \pi[$. Furthermore, define

$$\delta_k = u(d_k^+) - u(d_k^-), \quad k = 1, 2, \ldots, n.$$

To obtain a function continuous on $[-\pi, \pi]$, we consider the following construction:

$$w(x) = u(x) + \sum_{k=1}^{n} \frac{\delta_k}{2\pi} v(x + \pi - d_k), \quad x \in [-\pi, \pi].$$



Since $v(x)$ has a jump at $x = \pm\pi + 2\pi k$ of the size 2π (jumps from π to $-\pi$), it is clear that

$$\begin{aligned} w(d_m^+) - w(d_m^-) &= \delta_m + \frac{\delta_m}{2\pi} v(d_m^+ + \pi - d_m) - \frac{\delta_m}{2\pi} v(d_m^- + \pi - d_m) \\ &= \delta_m \left(1 + \frac{v(d_m^+ + \pi - d_m) - v(d_m^- + \pi - d_m)}{2\pi}\right) = \delta_m \left(1 + \frac{-\pi - \pi}{2\pi}\right) = 0 \end{aligned}$$

for $m = 1, 2, \ldots, n$. If $x \neq d_k$ for every $k = 1, 2, \ldots, n$, then w is continuous since both u and $x \mapsto v(x + \pi + d_k)$ are continuous at x. After possible redefinition at the points $\{d_k\}$, we have shown that w is continuous on $[-\pi, \pi]$ and that $w(\pi) = w(-\pi)$.
The previous theorem then proves that the Fourier series of w converges uniformly on $[-\pi, \pi]$. Moreover, we know that $v(x) = x$ has a Fourier series that converges uniformly on $[-c, c]$ for any $0 < c < \pi$. This implies that the Fourier series of $\frac{\delta_k}{2\pi} v(x + \pi - d_k)$ converges uniformly on any interval $[a, b] \subset]-\pi, \pi[$ that does not contain d_k. By assumption, u is continuous on $[a, b]$ so there are no jump points in $[a, b]$. Since the Fourier series of both $w(x)$ and $\sum_{k=1}^{n} \frac{\delta_k}{2\pi} v(x + \pi - d_k)$ converges uniformly on $[a, b]$, this must also hold for the Fourier series of $u(x)$ on $[a, b]$. □

4.4 Periodic Solutions to Differential Equations

Remember the most fun part in TATA42? Yeah, me too! Suppose that we wish to find a periodic solution to a differential equation, could we make an ansatz and find a solution as its Fourier series? Hypothetically yes, but the theory is a bit more difficult and subtle than the corresponding case with a power series approach. Let's consider an example.
Notice first that if y is differentiable and periodic, then y' is also periodic with the same period. Indeed,

$$y'(x + T) = \lim_{h \to 0} \frac{y(x + T + h) - y(x + T)}{h} = \lim_{h \to 0} \frac{y(x + h) - y(x)}{h} = y'(x).$$

Example

Find all 2π-periodic solutions to $y''(x) - 2y(x+\pi) = \cos x$.

Solution. So, the plan is to assume that $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ and plug this into the equation and identify the coefficients c_k (just as in TATA42). Note though, that we expressed that $y(x)$ was *equal* to its Fourier series above. This is not clear without motivation, so here goes. Reformulating the equation, we see that $y'' = \cos x + 2y(x+\pi)$. Since we're seeking a function that's at least differentiable, this means that y must be continuous. Hence y'' is also continuous. Why? Well,

$$y'' = 2y(x+\pi) + \cos x \tag{4.1}$$

so since both y and $\cos x$ are continuous, this must mean that y'' is also continuous. This means that $y \in C^2(\mathbf{R})$. Therefore, the right-hand side of (4.1) is obviously twice differentiable and so y''' must be continuous. Hence $y \in C^3(\mathbf{R})$. This is sufficient for letting

$$\begin{aligned}
y(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \\
y'(x) &= \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}, \\
y''(x) &= \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx},
\end{aligned}$$

something that is clear from Dirichlet's theorem (if $f \in E'$ is continuous then the Fourier series converges to f). Furthermore,

$$y(x+\pi) = \sum_{k=-\infty}^{\infty} c_k e^{ik(x+\pi)} = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi} e^{ikx} = \sum_{k=-\infty}^{\infty} c_k (-1)^k e^{ikx}.$$

Therefore, we must have

$$y''(x) - 2y(x+\pi) = \cos x \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 - 2(-1)^k) c_k e^{ikx} = \cos x = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}.$$

By uniqueness (we're looking for continuous functions), it then follows that

$$\begin{aligned}
c_k(-k^2 - 2(-1)^k) &= 1/2, \quad k = \pm 1, \\
c_k(-k^2 - 2(-1)^k) &= 0, \quad k \neq \pm 1.
\end{aligned}$$

So $c_1 = 1/2$ and $3c_{-1} = 1/2$, so $c_{-1} = c_1 = 1/2$. For $k \neq \pm 1$, we must have $c_k = 0$ or $-k^2 - 2(-1)^k = 0$. Clearly

$$-k^2 - 2(-1)^k = 0 \quad \Leftrightarrow \quad k^2 = 2(-1)^{k+1}$$

has no solutions in $\mathbf{Z}$ since either $k^2 = -2$ (nothing real) or $k^2 = 2$ (nothing integer valued or even rational). We have now obtained that

$$y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix} = \cos x.$$

4.5 Rules for Calculating Fourier Coefficients

Linearity

Theorem. Suppose that $u, v \in E$. Then $\widehat{au + bv}[k] = a\widehat{u}[k] + b\widehat{v}[k]$.

Proof. This follows from the linearity of the integral and the fact that everything is convergent for functions in E. □

We've already seen the following result in the previous sections.

Differentiation

Theorem. Suppose that $u' \in E$ and u is continuous with $u(-\pi) = u(\pi)$. If $u \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$, then $u'(x) \sim \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}$. That is, $\widehat{u'}[k] = ik\widehat{u}[k]$.

Example

Let $u(x) = |x|$ for $-\pi \le x \le \pi$. Use the fact that $u'(x) = \operatorname{sgn}(x)$ for $0 < |x| < \pi$ to find the Fourier series of $\operatorname{sgn}(x)$.

Solution. Recall that

$$u(x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx},$$

and that we have equality since $u \in E'$ and u is continuous. The Fourier coefficients are $c_0 = \pi/2$ and $c_k = \dfrac{(-1)^k - 1}{\pi k^2}$ for $k \neq 0$. Noting that $u'(x) = -1$ if $-\pi < x < 0$ and $u'(x) = 1$ if $0 < x < \pi$, we see that $u'(x) = \operatorname{sgn}(x)$ when $0 < |x| < \pi$. Hence $\operatorname{sgn}(x)$ has the Fourier coefficients $d_0 = 0$ and

$$d_k = ikc_k = ik\frac{(-1)^k - 1}{\pi k^2} = i\frac{(-1)^k - 1}{\pi k}, \quad k \neq 0,$$

and thus the Fourier series

$$\operatorname{sgn}(x) \sim \frac{i}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^k - 1}{k}.$$

This is true since $u' \in E$, $u(-\pi) = u(\pi)$ and u is continuous. Observe also that it doesn't matter that $u'(x) \neq \operatorname{sgn}(x)$ at some points. In fact, as long as the set of points where $u'(x) \neq \operatorname{sgn}(x)$ is small enough that it doesn't affect the integration, we'll obtain the same Fourier series. This is true for any Fourier series calculation. However, what the Fourier series converges to might not be the function at these exception points.

Mirroring

Theorem. Suppose that $u \in E$. Then $\widehat{u(-x)}[k] = \widehat{u(x)}[-k]$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt = /s = -t/ = -\int_{\pi}^{-\pi} u(s)e^{iks}\,ds = \int_{-\pi}^{\pi} u(s)e^{-i(-k)s}\,dt = 2\pi c_{-k}.$$

Conjugation

Theorem. Suppose that $u \in E$. Then $\widehat{\overline{u(x)}}[k] = \overline{\widehat{u}[-k]}$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} \overline{u(t)}e^{-ikt}\,dt = \int_{-\pi}^{\pi} \overline{u(t)e^{ikt}}\,dt = \overline{\int_{-\pi}^{\pi} u(t)e^{-i(-k)t}\,dt} = 2\pi\overline{c_{-k}}.$$

Translation

Theorem. Suppose that $u \in E$. Then $\widehat{u(x-y)}[k] = e^{-iky}\widehat{u}[k]$ for $k \in \mathbf{Z}$.

Proof:

$$\int_{-\pi}^{\pi} u(x-y)e^{-ikx}\,dx = /\ t = x-y\ / = \int_{-\pi-y}^{\pi-y} u(t)e^{-ik(t+y)}\,dt = e^{-iky}\int_{-\pi-y}^{\pi-y} u(t)e^{-ikt}\,dt$$
$$= /\ u \text{ is } 2\pi\text{-periodic}\ / = e^{-iky}\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt = e^{-iky}2\pi c_k.$$

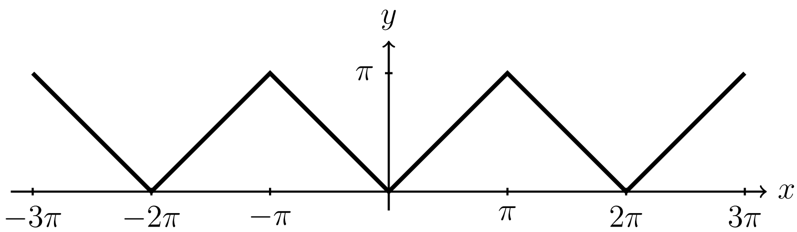
Example

Let $u(x) = |x|$ for $-\pi \le x \le \pi$ and periodically extend u. Find the Fourier coefficients for $u(x-1)$. To what does the Fourier series converge? What is the Fourier series for the function $w(x) = |x-1|$, $-\pi \le x \le \pi$?

Solution. This is a good example since it shows the dangers of not remembering that we're working with periodic extensions outside the domain $[-\pi, \pi]$.
Recall that

$$u(x) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k - 1}{\pi k^2} e^{ikx}.$$

Drawing the function looks like this.



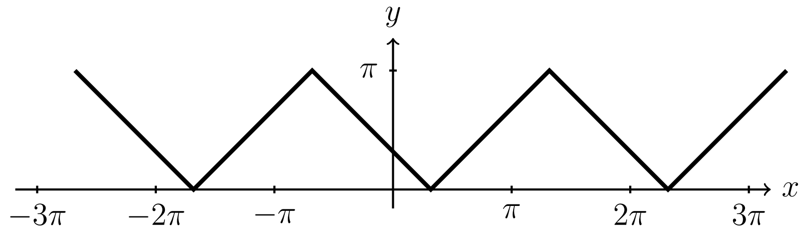
It is a periodic function. Now, finding the Fourier coefficients for $u(x-1)$ is rather easy if we use the "rule" above:

$$\widehat{u(x-1)}[k] = e^{-ik}\widehat{u}[k] = e^{-ik}\frac{(-1)^k - 1}{\pi k^2}, \quad k \neq 0,$$

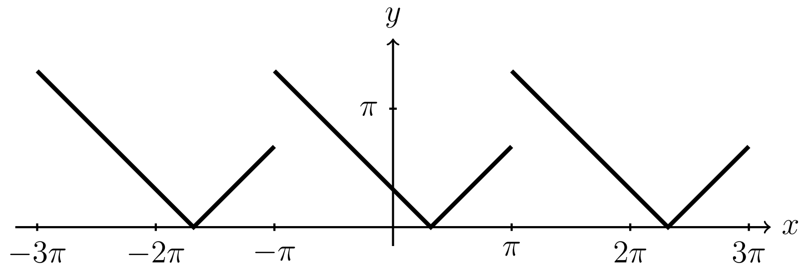
and $\widehat{u(x-1)}[0] = e^{-i\cdot 0}\frac{\pi}{2} = \frac{\pi}{2}$. This means that

$$u(x-1) = \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{-ik}\frac{(-1)^k - 1}{\pi k^2}e^{ikx},$$

again with equality since $u(x-1)$ is a continuous function in E'. Note now though what the graph looks like.



It is a shifted copy of the graph of $u(x)$, which was something that we periodically extended. If we *actually* want to find the Fourier series for $w(x) = |x-1|$, $-\pi \leq x \leq \pi$, we have to do a new calculation and this would look different. Furthermore, the Fourier series will not converge to $|x-1|$ at odd multiples of π. Drawing what $w(x)$ actually looks like (and periodically extend w) makes this clear.



Doing the calculation, you would find that

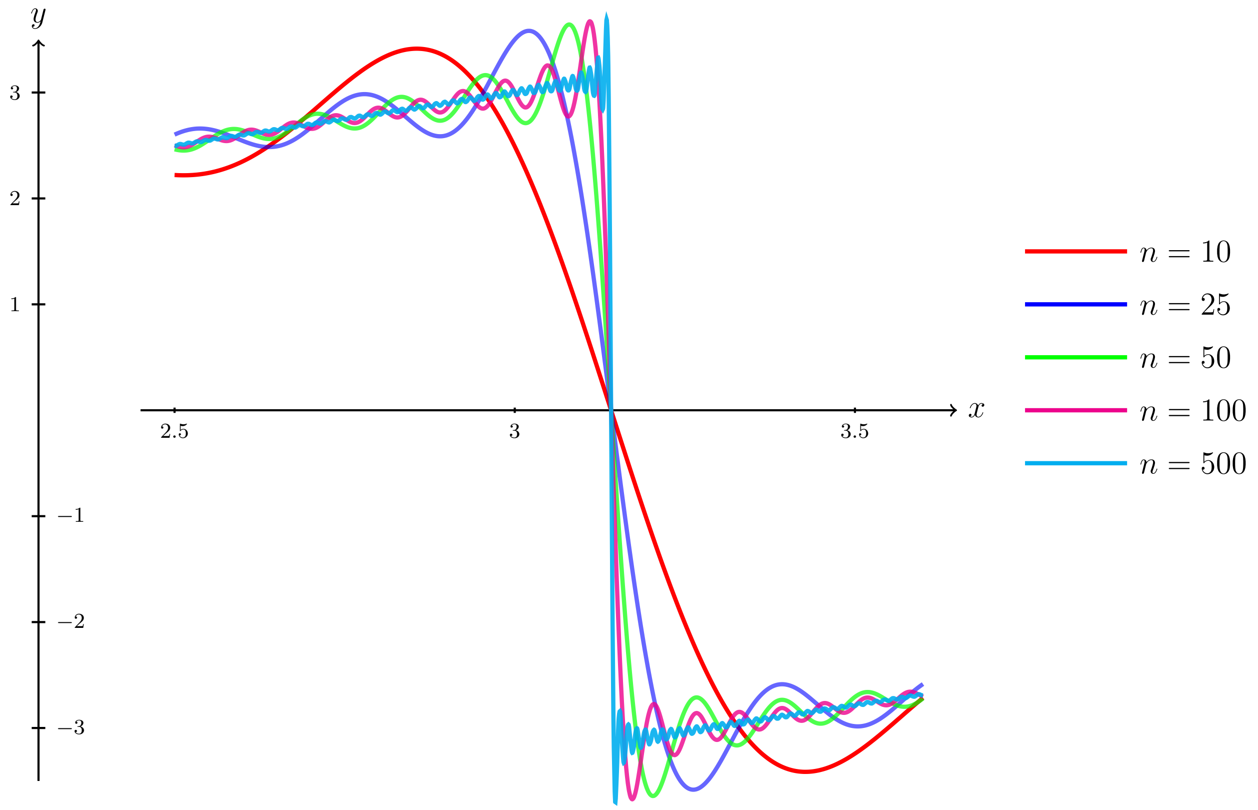
$$w(x) \sim \frac{1+\pi^2}{2\pi} + \frac{1}{\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k(1-ik) - e^{-ik}}{k^2}.$$

When is $w(x)$ equal to it's Fourier series?

4.6 Gibbs' Phenomenon

As we saw in the first example of this lecture, the Fourier series of $v(x) = x$ behaves rather odd at the points $x = \pm\pi$. We've seen this squiggly behavior previously as well. For example when looking at the Fourier series for $\mathrm{sgn}(x)$ we saw that this happened at the origin. However, the function $x \mapsto x^2$ did not exhibit the oscillatory stuff. Why? The reason is the continuity. We've seen in this lecture that we have uniform convergence on closed intervals under rather generous conditions that include continuity of the function, and what is common with all the squiggly sums is that the functions has a jump around which the Fourier partial sums oscillate. Let's take a zoomed in look at the partial sums for $v(x) = x$, that is

$$T_m(x) = \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \sin kx.$$



What is very interesting in this picture is that the height of the oscillations (at the extremes) seems to be the same no matter how many terms we use in the partial sum. What changes is that the oscillations gets more squeezed together around the jump point. Now this is just a specific example, but it turns out that this holds for *all* functions with jump discontinuities. The height of the wobbliness is about 9% of the size of the jump. This is known as Gibbs' phenomenon.

To see why this holds in this case (which we will use to show the general case below), consider the sequence $\{x_m\}$ defined by $x_m = \pi(1 - m^{-1})$. Then

$$\begin{aligned}
T_m(x_m) &= \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \sin\left(k\pi\left(1 - \frac{1}{m}\right)\right) \\
&= \sum_{k=1}^{m} \frac{2(-1)^{k+1}}{k} \left(\sin k\pi \cos\left(\frac{\pi}{m}\right) - \cos k\pi \sin\left(\frac{k\pi}{m}\right)\right) \\
&= \sum_{k=1}^{m} \frac{2(-1)^{2k+2}}{k} \sin\left(\frac{k\pi}{m}\right) = \sum_{k=1}^{m} \frac{2}{k} \sin\left(\frac{k\pi}{m}\right) = 2\sum_{k=1}^{m} \frac{\sin\left(\frac{k\pi}{m}\right)}{\frac{k\pi}{m}} \cdot \frac{\pi}{m}.
\end{aligned}$$

Next we observe that this is a Riemann sum for the function $x \mapsto \dfrac{\sin x}{x}$ on the interval $[0,\pi]$, so since this function is Riemann integrable, letting $m \to \infty$ yields that

$$\lim_{m\to\infty} T_m(x_m) = 2\int_0^\pi \frac{\sin x}{x}\,dx \geq 1.18\pi.$$

Furthermore, certainly $v(x_m) \to \pi$ increasingly as $m \to \infty$ and $v(\pi^+) - v(\pi^-) = -2\pi$, so the size of the jump is 2π. Therefore,

$$\lim_{m\to\infty} \frac{T_m(x_m) - v(x_m)}{2\pi} \approx \frac{1.18\pi - \pi}{2\pi} = 0.09,$$

so for m large enough, we have

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \geq 0.089.$$

This means that there's a sequence $\{x_m\}$ for which the difference between $T_m(x_m)$ and $v(x_m)$ is about 9% of the the size of the jump!
To summarize, we have a sequence $\{x_m\}$ such that $x_m < \pi$ and $x_m \to \pi^-$ increasingly and for which

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \geq 0.089,$$

for m large enough. Completely analogously, there exists a sequence $\{x_m\}$ such that $x_m > \pi$ and $x_m \to \pi^+$ decreasingly and for which

$$\frac{T_m(x_m) - v(x_m)}{2\pi} \leq -0.089$$

for m large enough.

What's even more impressive is that this fact holds for all jumps when dealing with functions from E. The size of the oscillations are always about 9% of the size of the jump. So large jumps (like at the endpoints) cause a lot of "overshooting" where a signal will look weird (and the amplitude will overshoot the "expected" signal). To formalize this, we have the following theorem.

Theorem. Suppose that $u \in E$ is continuous on the interval $[d-\delta, d+\delta]$ except at $x = d$ and suppose that $u' \in E$. Let $\delta_d = u(d^+) - u(d^-)$. Then there exists a sequence $x_m \to d^+$ such that

$$\lim_{m\to\infty} \frac{S_m(x_m) - u(x_m)}{\delta_d} \geq 0.089,$$

where $S_m(x)$ are the partial Fourier sums of u.

Proof. Using the same idea that was used when proving uniform convergence on sub-intervals, let us define the function w by

$$w(x) = u(x) + \frac{\delta_d}{2\pi} v(x + \pi - d),$$

where $v(x) = x$ for $-\pi < x < \pi$ and $v(\pm\pi) = 0$. Then w is continuous on $I = [d - \delta, d + \delta]$ and since $w, w' \in E$, the Fourier series of w converges uniformly on I. Thus, for any $\epsilon > 0$, we may assume that $\delta > 0$ is small enough such that

$$|W_m(x) - w(x)| < |\delta_d|\epsilon$$

for $m \geq N$ (some $N \in \mathbf{Z}$) and all $x \in I$, where $W_m(x)$ is the partial Fourier series of $w(x)$. Since

$$W_m(x) = S_m(x) + \frac{\delta_d}{2\pi} T_m(x + \pi - d), \quad m = 1, 2, 3, \ldots,$$

it follows that

$$\begin{aligned}\frac{S_m(x) - u(x)}{\delta_d} &= \frac{W_m(x) - \frac{\delta_d}{2\pi} T_m(x + \pi - d) - (w(x) - \frac{\delta_d}{2\pi} v(x + \pi - d))}{\delta_d} \\ &= \frac{W_m(x) - w(x)}{\delta_d} - \frac{T_m(x + \pi - d) - v(x + \pi - d)}{2\pi}.\end{aligned}$$

By the argument above, there is a sequence $x_m > d$ such that

$$\frac{T_m(x + \pi - d) - v(x + \pi - d)}{2\pi} \leq -0.089,$$

from which it is clear that

$$\lim_{m \to \infty} \frac{S_m(x_m) - u(x_m)}{\delta_d} \geq 0.089,$$

since ϵ was arbitrary. □

Chapter 5

Uniqueness, Convergence in Mean, Completeness

"See you at the party, Richter."
—Douglas Quaid (Hauser)

5.1 Uniqueness

So we have seen conditions when the Fourier series of a function u converges (and to what). Another important question is in what sense we can expect the Fourier coefficients to represent a given function.
Question. Suppose that $u, v \in E$ has the Fourier series'

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{and} \quad v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}.$$

If $c_k = d_k$ for every $k \in \mathbf{Z}$, what can we say about u and v?
We know from before that if $u, v \in E'$ are continuous, then the Fourier series' converge to u and v respectively, so if the Fourier coefficients are the same then the functions are equal. This is *not* true in general, but we will show that equality holds at points where both u and v are continuous. To approach this, we need some summation results.

5.1.1 Cesàro Summation

Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of numbers and let $S_n = \sum_{k=1}^{n} a_k$ denote the partial sums. We define

$$\overline{S}_n = \frac{1}{n} \sum_{l=1}^{n} S_l, \quad n = 1, 2, 3, \ldots,$$

to be the mean value of the first n partial sums of the sequence. So yeah, this is a sum of sums. If

$$\lim_{n \to \infty} \overline{S}_n = A$$

exists (in the usual convergent sense), then we say that the sequence $a_1, a_2, a_3, \ldots$ is Cesàro-summable. Note in particular that if $\sum_{k=1}^{\infty} a_k = S$ is convergent, then $A = S$, so we obtain the same answer when doing Cesàro summation. One can see this by considering the following. Let $S_n \to S$ be convergent and let $\epsilon > 0$. Then there exists $N \in \mathbf{Z}$ such that $|S_m - S| \leq \epsilon$ if $m \geq N$ and

$$|\overline{S}_n - S| = \left|\frac{1}{n}\sum_{k=1}^{n}(S_k - S)\right| \leq \frac{1}{n}\sum_{k=1}^{m}|S_k - S| + \frac{1}{n}\sum_{k=m+1}^{n}|S_k - S| \leq \frac{1}{n}\sum_{k=1}^{m}|S_k - S| + \epsilon \to \epsilon,$$

as $n \to \infty$. Thus $\overline{S}_n \to S$ as $n \to \infty$.
So why introduce this type of summing? Well, it makes it possible to assign values to series that are classically divergent.

Example
Is the sequence $1, -1, 1, -1, 1, -1, \ldots$ Cesàro summable?

Solution. The sequence is obviously not summable in the classical sense (why?). However, the answer to the question is yes. Consider the partial sums S_n. If n is even, then $S_n = 0$, and if n is odd, then $S_n = 1$. Since

$$\overline{S}_n = \frac{1}{n}\sum_{l=1}^{n} S_l,$$

we obtain that

$$\overline{S}_1 = 1, \quad \overline{S}_2 = \frac{1}{2}, \quad \overline{S}_3 = \frac{2}{3}, \quad \overline{S}_4 = \frac{2}{4} = \frac{1}{2}, \quad \overline{S}_5 = \frac{3}{5}, \quad \overline{S}_6 = \frac{3}{6} = \frac{1}{2}, \quad \overline{S}_7 = \frac{4}{7}, \quad \overline{S}_8 = \frac{4}{8} = \frac{1}{2},$$

and so on. Thus $\overline{S}_{2k} = 1/2$ and $\overline{S}_{2k+1} \to 1/2$ as $n \to \infty$, so $\overline{S} = 1/2$.
Note: this special series is usually referred to as *Grandi's series*.

5.1.2 The Fejér Kernel

Let us look at what happens if we try to perform Cesàro summation for a Fourier series. Working with the complex Fourier series, we define

$$\overline{S}_n(x) = \frac{1}{n+1}\sum_{l=0}^{n} S_l(x) = \frac{S_0(x) + S_1(x) + \cdots + S_n(x)}{n+1},$$

where

$$S_l(x) = \sum_{k=-l}^{l} c_k e^{ikx}, \quad l = 0, 1, 2, \ldots,$$

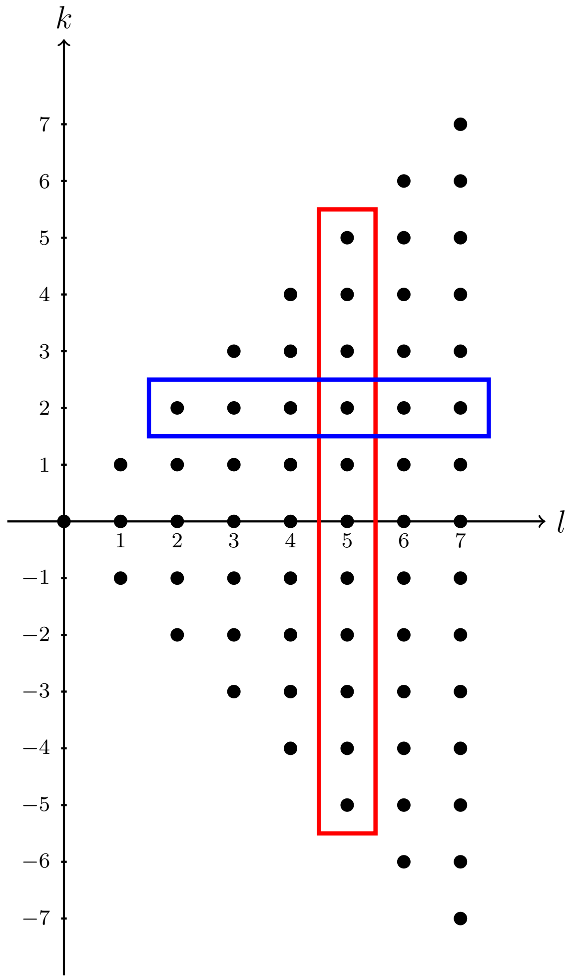
and c_k are the complex Fourier coefficients. The expression for $\overline{S}_n(x)$ is basically the Cesàro mean for the symmetric partial sums. Let us proceed like we did when identifying the Dirichlet kernel

$$\begin{aligned}\overline{S}_n(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} c_k e^{ikx} = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} u(t)e^{-ikt}\,dt\right)e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\left(\frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l} e^{ik(x-t)}\right)dt.\end{aligned}$$

Isolating the inner parenthesis, we notice that

$$\begin{aligned}\frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l}e^{ik(x-t)} &= \frac{1}{n+1}\sum_{k=-n}^{n}e^{ik(x-t)}\sum_{l=|k|}^{n}1 = \sum_{k=-n}^{n}\frac{n-|k|+1}{n+1}e^{ik(x-t)}\\ &= \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ik(x-t)},\end{aligned}$$

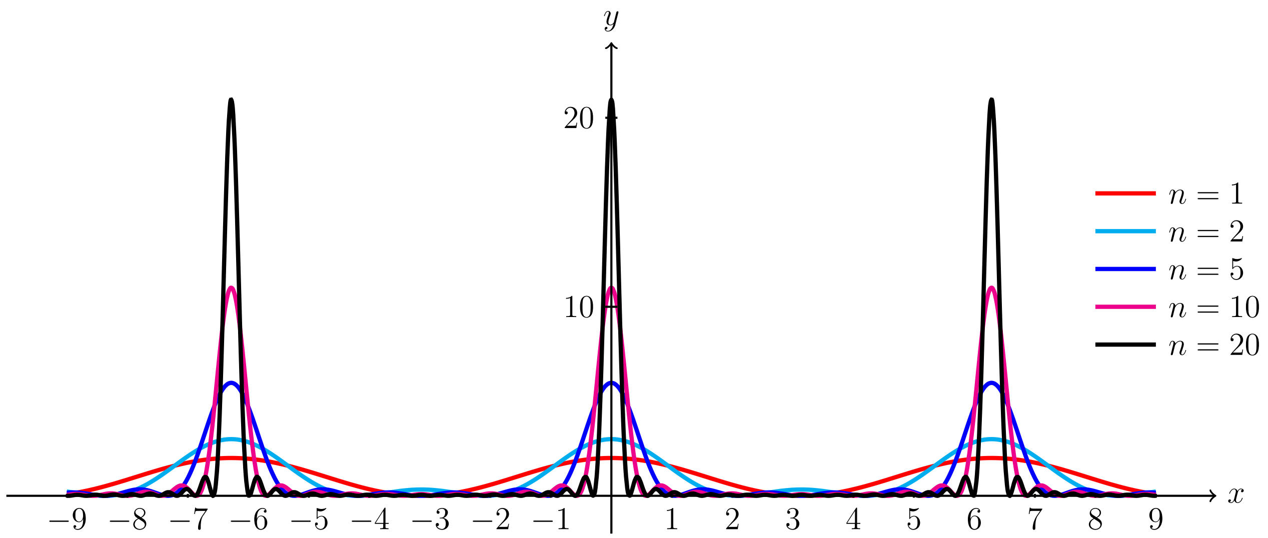
where we changed the order of summation. To see why this looks the way it does, consider the figure below (it's the same type of thinking like we did with multiple integrals). Instead of summing over the red rectangles we switch and sum over the blue ones instead.



The Fejér kernel

Definition. We define the **Fejér kernel** $F_n(x)$ as

$$F_n(x) = \frac{1}{n+1}\sum_{l=0}^{n}\sum_{k=-l}^{l}e^{ikx} = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}, \quad n = 0, 1, 2, \ldots$$



Obviously $F_n(x)$ is an even 2π-periodic function (similar to the Dirichlet kernel) and we can write

$$\overline{S}_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)F_n(x-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t+x)F_n(t)\,dt.$$

Notice also that $F_n(x)$ is a sum of Dirichlet kernels $D_l(x)$, giving rise to the representation

$$F_n(x) = \frac{1}{n+1}\sum_{l=0}^{n}\frac{\sin((2l+1)x/2)}{\sin(x/2)},\ x \neq 2k\pi,\ k \in \mathbf{Z}.$$

Properties of the Fejér kernel

Theorem.

(i) $F_n(2k\pi) = n+1,\ k \in \mathbf{Z}$.

(ii) $F_n(x) = \dfrac{1}{n+1}\left(\dfrac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2,\ x \neq 2k\pi,\ k \in \mathbf{Z}$.

(iii) $\displaystyle\int_T F_n(x)\,dx = 2\pi$.

(iv) If $0 < \tau < \pi$, then $F_n \to 0$ uniformly on the set $[-\pi, -\tau] \cup [\tau, \pi]$ as $n \to \infty$.

Proof. We obtain the first point by direct verification from the definition of F_n. To prove the second identity, observe that

$$\begin{aligned}(n+1)\sin(x/2)^2 F_n(x) &= -\frac{1}{4}\sum_{l=0}^{n}\left(e^{ix/2}-e^{-ix/2}\right)\left(e^{i(2l+1)x/2}-e^{-i(2l+1)x/2}\right)\\ &= -\frac{1}{4}\sum_{l=0}^{n}\left(e^{i(l+1)x}-e^{-ilx}-e^{ilx}+e^{-i(l+1)x}\right)\\ &= -\frac{1}{4}\sum_{l=0}^{n}\left(2\cos(l+1)x-2\cos lx\right) = /\text{ telescoping sum }/\\ &= -\frac{1}{2}\left(\cos(n+1)x-\cos 0\right) = \frac{1-\cos(n+1)x}{2}\\ &= \sin^2\left((n+1)x/2\right).\end{aligned}$$

Furthermore, we see that

$$\int_{-\pi}^{\pi} F_n(x)\,dx = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\int_{-\pi}^{\pi} e^{ikx}\,dx = 2\pi,$$

since e^{ikx} is 2π-periodic when $k \in \mathbf{Z}$ and $k \neq 0$.
The last point is a little more subtle. Looking at the graphs above, we see that the mass seems to be centering more and more around the origin, so we might expect something if we avoid the origin. Indeed, we can see that

$$\|F_n\|_{C[\tau,\pi]} = \frac{1}{n+1}\max_{\tau\le x\le\pi}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2 \le \frac{1}{n+1}\max_{\tau\le x\le\pi}\frac{1}{\sin^2(x/2)} \le \frac{1}{n+1}\frac{1}{\sin^2(\tau/2)} \to 0,$$

so $F_n \to 0$ uniformly on $[\tau,\pi]$. This also implies uniform convergence for $[-\pi,-\tau]$ since F_n is an even function.

Theorem. Suppose that $u \in E$. Then

$$\lim_{n\to\infty}\overline{S}_n = \frac{u(x^+)+u(x^-)}{2}$$

for $x \in [-\pi,\pi]$.

Proof. This mirrors the proof of the corresponding theorem for $u \in E'$ when we used the Dirichlet kernel. We need to show that

$$\frac{1}{2\pi}\int_0^{\pi}(u(x+t)-u(x^+))F_n(t)\,dt + \frac{1}{2\pi}\int_{-\pi}^{0}(u(x+t)-u(x^-))F_n(t)\,dt \to 0,$$

as $n\to\infty$. This implies that

$$\lim_{n\to\infty}\overline{S}_n = \frac{u(x^+)+u(x^-)}{2}$$

since

$$\frac{1}{2\pi}\int_{-\pi}^{0}F_n(t)\,dt = \frac{1}{2\pi}\int_0^{\pi}F_n(t)\,dt = \frac{1}{2}.$$

Let $\epsilon > 0$. Since u has a right-hand limit at x, there is a $\delta > 0$ such that

$$0 < t < \delta \quad \Rightarrow \quad |u(x+t) - u(x^+)| < \epsilon.$$

We exploit this and the uniform convergence of F_n to obtain that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_0^\pi (u(x+t) - u(x^+))F_n(t)\,dt\right| &\leq \frac{1}{2\pi}\int_0^\delta \epsilon F_n(t)\,dt + \frac{1}{2\pi}\int_\delta^\pi |u(x+t) - u(x^+)|F_n(t)\,dt \\ &\leq \frac{\epsilon}{2}\frac{1}{\pi}\int_0^\pi F_n(t)\,dt + \frac{1}{2\pi}\int_\delta^\pi |u(x+t) - u(x^+)|F_n(t)\,dt \\ &\to \frac{\epsilon}{2}\end{aligned}$$

as $n \to \infty$ since F_n converges uniformly to zero on $[\delta, \pi]$. The second integral is handled analogously. □

The following corollary is clear since if $S_n(x)$ converges, then $\overline{S}_n(x)$ converges to the same value.

Corollary. Suppose that $u \in E$. If the Fourier series is convergent at $x \in [-\pi, \pi]$, then

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{u(x^+) + u(x^-)}{2}.$$

So basically we could say that "if it converges, it converges correctly" (where it refers to the Fourier series of something in E).

Corollary. Suppose that $u, v \in E$. If

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad \text{and} \quad v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}$$

and $c_k = d_k$ for every $k \in \mathbf{Z}$, then $u(x) = v(x)$ at every point $x \in [-\pi, \pi]$ where both u and v are continuous.

If u is continuous on $[-\pi, \pi]$, then we can use the uniform continuity of u in the proof above to show that $\overline{S_n}(x)$ converges uniformly (for a fixed ϵ we can use the same δ for every x).

Uniform convergence

Corollary. If $u \in E$ is continuous and $u(-\pi) = u(\pi)$, then $\overline{S_n}(x)$ converges uniformly to u.

5.2 E, E' and All That Stuff

So we've seen results now that requires different things of the function u to obtain convergence in different senses. To summarize, some of the things we know are the following.

(i) If $u \in E$, then u has a Fourier series (convergence of which is unknown).

(ii) If $u \in E$, then $\overline{S_n}(x) \to \dfrac{u(x^+) + u(x^-)}{2}$.

(iii) If $u, v \in E$ and $\widehat{u}[k] = \widehat{v}[k]$, $k \in \mathbf{Z}$, then $u(x) = v(x)$ whenever u and v are continuous at x.

(iv) If $u \in E$ and $D^{\pm}u(x)$ exists, then $S_n(x) \to \dfrac{u(x^+) + u(x^-)}{2}$. If $u \in E'$, this limit holds for all x.

(v) If $u' \in E$, u is continuous and $u(-\pi) = u(\pi)$, then $S_n(x)$ converges uniformly to $u(x)$.

(vi) If $u' \in E$ and u is continuous on $[a, b] \subset]-\pi, \pi[$, then $S_n(x)$ converges uniformly on $[a, b]$.

It is therefore reasonable to question as to whether there are differences between these classes of functions. First, let's take a look at the one-sided derivatives.

Theorem. If $u' \in E$, then $D^{\pm}u(x) = \lim_{y \to x^{\pm}} u'(y)$.

Proof. If $u' \in E$, then u' is piecewise continuous. If x is a point of continuity for u', then $D^{\pm}u(x) = u'(x)$ immediately. If x is a "jump"-point for u', we need to be a bit more careful. Let $h > 0$ and recall the mean value theorem: if u is continuous on $[x, x+h]$ and differentiable on $]x, x+h[$, then there exists a number ξ such that

$$\frac{u(x+h) - u(x)}{h} = u'(\xi), \quad \text{where } x < \xi < x + h.$$

Letting $h \to 0^+$, we find that

$$D^+u(x) = \lim_{h \to 0^+} \frac{u(x+h) - u(x)}{h} = \lim_{h \to 0^+} u'(\xi) = u'(x^+),$$

since $u' \in E$ and $\xi \to x^+$. The left-hand derivative $D^-u(x)$ is handled analogously. $\square$

The previous theorem does *not* hold if we only know that $u \in E'$. If we don't know that the derivative is continuous on $]x - \delta, x[$ or $]x, x + \delta[$, then we have to use the definition of $D^-u(x)$ and $D^+u(x)$ directly.

The difference between $u' \in E$ and $u \in E'$

So what does $u' \in E$ mean? We intend for this to mean that the derivative is a piecewise continuous function. What this entails for u is that the two-sided derivative might not exist at some points, but we still write $u' \in E$. The reason for this is that we don't care what actually happens at individual points, but rather the limiting behavior of the function when we approach the point.
If $u \in E'$, then we only know that the function has one-sided derivatives at every point. This is *not* sufficient for u' to be piecewise continuous. In fact u' might be very discontinuous.

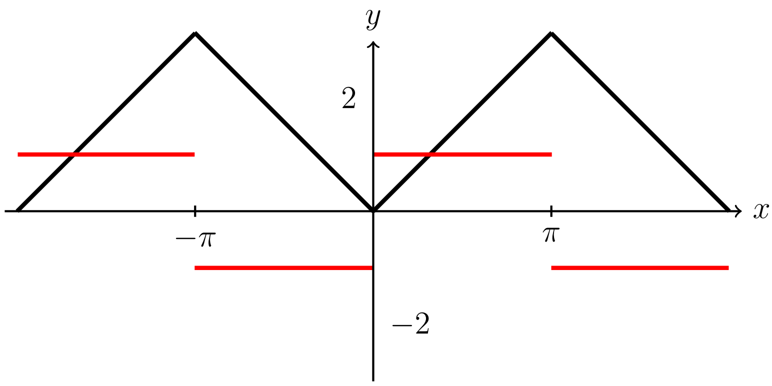
5.2.1 Some Examples

Let's consider some examples that show the differences between the conditions. The black graphs depict the function and the red graphs the derivative.

Example

Let $u(x) = |x|$ for $-\pi \leq x \leq \pi$ and extend periodically. Then $u \in E$ is continuous and $u \in E'$. Moreover, $u' \in E$.

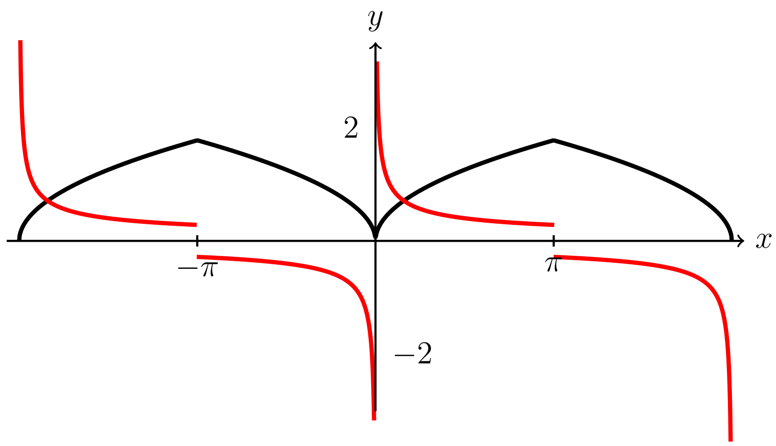
Clearly $u'(x) = -1$ if $x < 0$ and $u'(x) = 1$ if $x > 0$. At $x = 0$, $u'(0)$ does not exist. However, $D^{\pm}u(0) = \pm 1$.



Example

Let $u(x) = \sqrt{|x|}$ for $-\pi \leq x \leq \pi$ and extend periodically. Then $u \in E$ is continuous but $u \notin E'$.

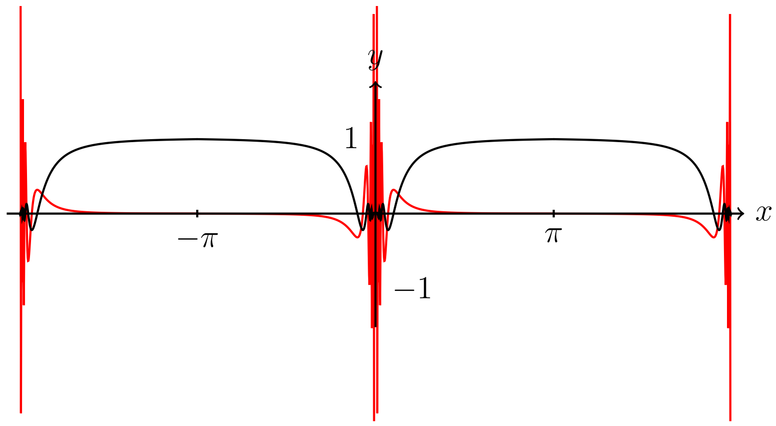
For $-\pi < x < 0$, we find that $u'(x) = -|x|^{-1/2}/2$ and $D^{-}u(0)$ doesn't exist (would be $-\infty$). However, $D^{+}u(-\pi) = -|\pi|^{1/2}/2$. Analogously, for $0 < x < \pi$, we find that $u'(x) = |x|^{-1/2}/2$ and $D^{+}u(0)$ doesn't exist (would be ∞). However, $D^{-}u(\pi) = |\pi|^{1/2}/2$.



Example

Let $u(x) = x \sin \frac{1}{x}$ for $-\pi \leq x \leq \pi$ and extend periodically. Then $u \in E$ is continuous but $u \notin E'$. In fact, $D^{\pm}u(0)$ does not even exist if $\pm\infty$ is allowed (this is worse than $\sqrt{|x|}$).

For $-\pi \leq x \leq \pi$ and $x \neq 0$, it is clear that $u'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$, but $D^{\pm}u(0)$ does not exist (not even if we allow $\pm\infty$ as possibilities). In the graph below, the scale of the function is ten times the size of the derivative.



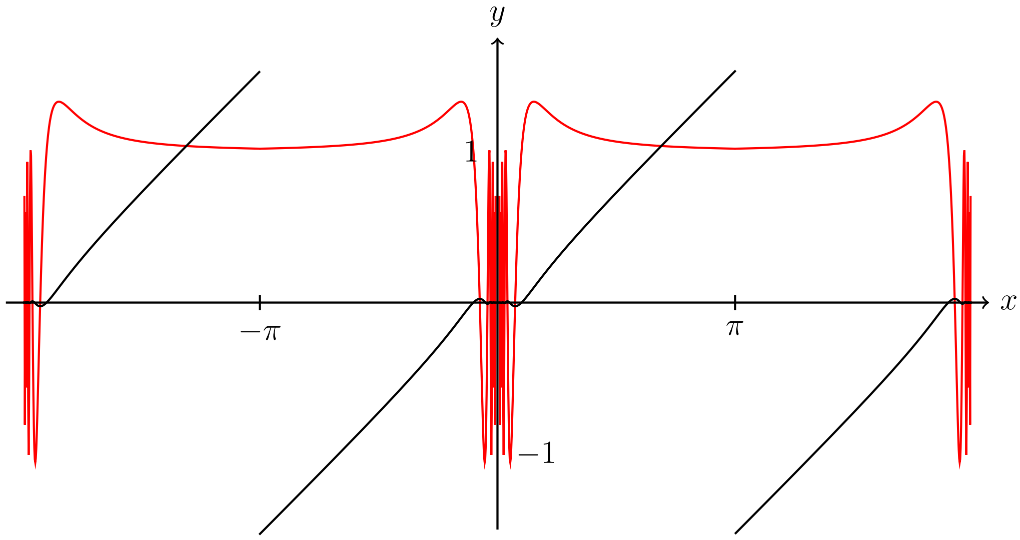
Example

Let $u(x) = x^2 \sin \frac{1}{x}$ for $-\pi \leq x < 0$ and $0 < x < \pi$. Put $u(0) = 0$ and extend u periodically. Then u' exists everywhere in $]-\pi, \pi[$ and $D^{\pm}u(-\pi)$ and $D^{\pm}u(\pi)$ exists. However, the derivative u' is discontinuous at $x = 0$. Moreover, $u' \notin E$.

For $-\pi < x < \pi$ and $x \neq 0$, it is clear that $u'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. For $x = 0$, we find that

$$u'(0) = \lim_{h \to 0} \frac{u(h) - u(0)}{h} = \lim_{h \to 0} \frac{1}{h} h^2 \sin \frac{1}{h} = 0 = \lim_{h \to 0} h \sin \frac{1}{h} = 0,$$

so $u'(0)$ exists. Clearly it is not true that $u'(x) \to 0$ as $x \to 0^{\pm}$, so by the theorem we proved earlier it is impossible that $u' \in E$. More directly, consider the limit of $u'(x)$ as $x \to 0^{\pm}$. Neither limit exists, so $u' \notin E$.



Example

The Weierstrass function $W(x)$ (look back at the section with the M-test in lecture 3) is a continuous function, so $W \in E$. However, this function is nowhere differentiable going so far that $|D^{\pm}W(x)| = \infty$ at every point. Clearly $W \notin E'$.

5.2.2 How Discontinuous Can a Derivative Be?

So the previous examples had problems at a single point (and maybe at the endpoints). Obviously we can construct something that has problems at any each point of any finite set (which would make the function look quite horrible), but from a mathematical perspective that's usually not that bad. Could we have problems at an infinite set of points?
Let's recall a famous theorem by Darboux, claiming that the derivative of a differentiable function has the intermediate value property.

Darboux's theorem

Theorem. Suppose that u is differentiable on $[a, b]$ and that $u'(a) < u'(b)$. If λ is a number such that $u'(a) < \lambda < u'(b)$, then there exists a point $c \in]a, b[$ such that $u'(c) = \lambda$.

Proof. We want to prove that there exists some $c \in]a, b[$ such that $u'(c) - \lambda = 0$. Let's define $U(x) = u(x) - \lambda x$ so that $U'(c) = u'(c) - \lambda$. Then $U'(a) = u'(a) - \lambda < 0$. Hence there's some point $x_0 > a$ such that $U(x_0) < U(a)$. Similarly, since $U'(b) = u'(b) - \lambda > 0$, there's some point $x_1 < b$ such that $U(x_1) < U(b)$.
What this means, is that the minimum of U on $[a, b]$ is *not* attained at the endpoints. With U being a continuous function and $[a, b]$ being compact, we do however know that the minimum is attained. This ensures the existence of a point $c \in]a, b[$ such that $U(c)$ is an extreme value and since U is differentiable, this proves that $U'(c) = 0$. □

So what's the use of this result? For one thing, we can show that certain functions *can't* be the derivative of something else. Indeed, as an example consider the function $u(x) = 1$ if x is irrational and $u(x) = 0$ if x is rational. This is a severely discontinuous function. Assuming

that u is the derivative of some function U, it would follow from Darboux's theorem that u has the intermediate value property. This is obviously false since we can choose any number $\lambda \in]0,1[$ where we can't find any c such that $u(c) = \lambda$.
So in other words, if a function is differentiable, then the derivative can't be as bad as this. However, there are differentiable functions where the set of of discontinuities of the derivative is uncountable so it's still pretty bad. In fact, there are functions whose derivative is so bad that you can't integrate the derivative with the Riemann integral.

5.3 The ON-system $\{e^{ikx}\}_{k\in\mathbf{Z}}$ is closed in E

We will now prove that the ON-system $\{e^{ikx}\}_{k\in\mathbf{Z}}$ is closed in E, meaning that we need to show that for every $u \in E$, there is a sequence of constants $c_k \in \mathbf{C}$, $k = 0, 1, 2, \ldots$ such that

$$\lim_{n\to\infty} \left\| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 = \lim_{n\to\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right|^2 dx \right)^{1/2} = 0. \tag{5.1}$$

Note that this result will imply that the Fourier series of $u \in E$ will converge to u in the sense of the norm we use on E (the L^2-norm). This is sometimes called *convergence in mean.*
To obtain this result, we need a sequence of approximation results rather typical for (hard) analysis. Recalling from the previous section that we can approximate any continuous function on $[-\pi, \pi]$ uniformly by the trigonometric polynomial that is its Fourier series, we need to first approximate $u \in E$ with something continuous.
The procedure will be as follows. We fix some $u \in E$. Next we choose a piecewise constant function h such that

$$\|u - h\|_2 < \frac{\epsilon}{3}.$$

Next we approximate this piecewise constant function h by a piecewise linear[1] continuous function f (satisfying $f(-\pi) = f(\pi)$) such that

$$\|h - f\|_2 < \frac{\epsilon}{3}.$$

Now, since f is continuous and $f' \in E$, we know that the Fourier series of f converges to f uniformly for every $x \in [-\pi, \pi]$. This means that we can choose N so that

$$\left\| f(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 < \frac{\epsilon}{3}, \quad \text{for } n > N,$$

if c_k are the Fourier coefficients of f. Finally, by the triangle inequality we have now obtained that

$$\left\| u(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 \leq \|u - h\|_2 + \|h - f\|_2 + \left\| f(x) - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

when $n \geq N$, which is precisely what (5.1) means.

[1] Piecewise linear means that the function is of the form $y = kx + m$ on each "piece."

5.3.1 Approximations...

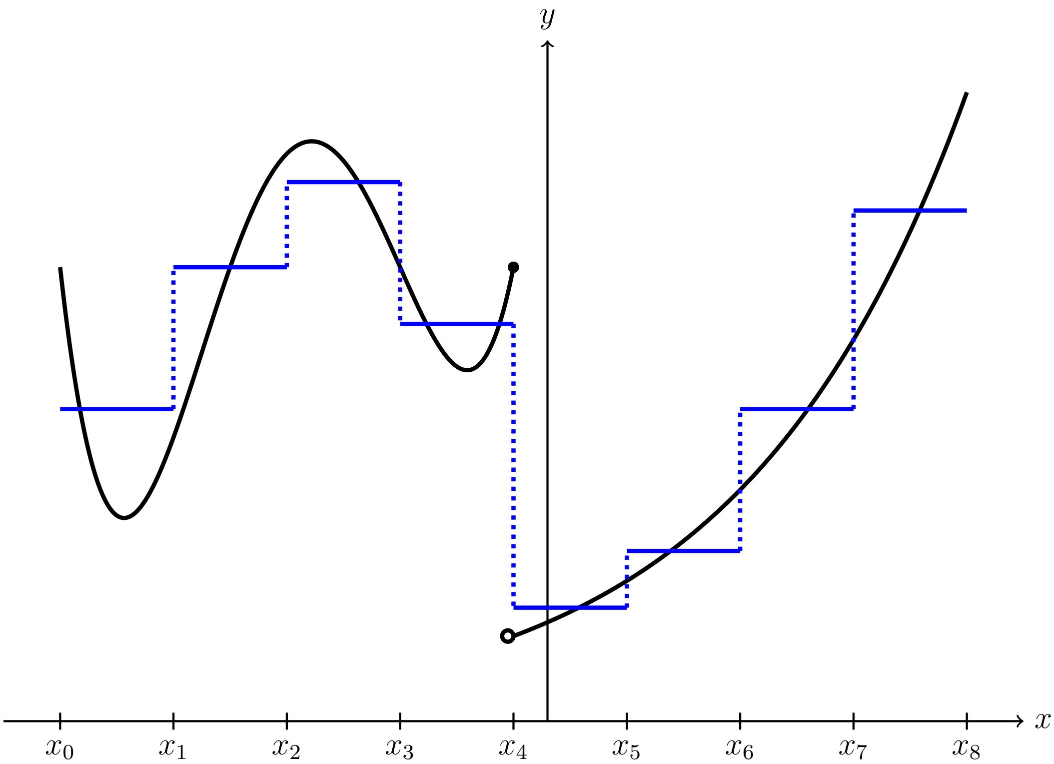
Let's take a closer look at these approximations.
First, since $u \in E$ it is Riemann integrable (see TATA41) and the following must hold. For every $\epsilon > 0$, there is a *partition* of $[-\pi, \pi]$,

$$x_0 = -\pi < x_1 < x_2 < \cdots < x_n = \pi,$$

and numbers $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(x) - h(x)|^2 \, dx < \frac{\epsilon^2}{4},$$

where we define the function $h(x)$ to be equal to $d_k = u(\xi_k)$ if $x_k < x \leq x_{k+1}$. Note that h is a piecewise constant function that approximates u. See the blue graph below.
We make sure to include the points where u is discontinuous (of which there are a finite number) in the set $\{x_0, x_1, \ldots, x_n\}$, so that u is continuous on each interval $[x_i, x_{i+1}]$ after possible redefinition at the endpoints (remember that the right- and lefthand limits of u exists if $u \in E$).



To see why this is possible, note that the restriction of u to intervals $[a_i, a_{i+1}]$ (after possible redefinition at a finite number of points a_i) is uniformly continuous on each $[a_i, a_{i+1}]$. Thus, for any $\epsilon > 0$, there is a $\delta_i > 0$ such that

$$x, y \in [a_i, a_{i+1}] : |x - y| < \delta_i \quad \Rightarrow \quad |u(x) - u(y)| < \frac{\epsilon}{3}.$$

Let $\delta = \min\{\delta_i\}$. Clearly $\delta > 0$, so it is possible to choose a partition $\{x_i\}_{i=0}^{n}$ of $[-\pi, \pi]$ such that $|x_{i+1} - x_i| < \delta$, $i = 0, 1, 2, \ldots, n-1$, and each point a_i can be found in the set $\{x_i\}_{i=0}^{n}$. By the uniform continuity on each $[x_i, x_{i+1}]$, it is clear that

$$|u(x) - h(x)|^2 = |u(x) - d_k|^2 \leq \frac{\epsilon^2}{9}, \quad x_i < x < x_{i+1},$$

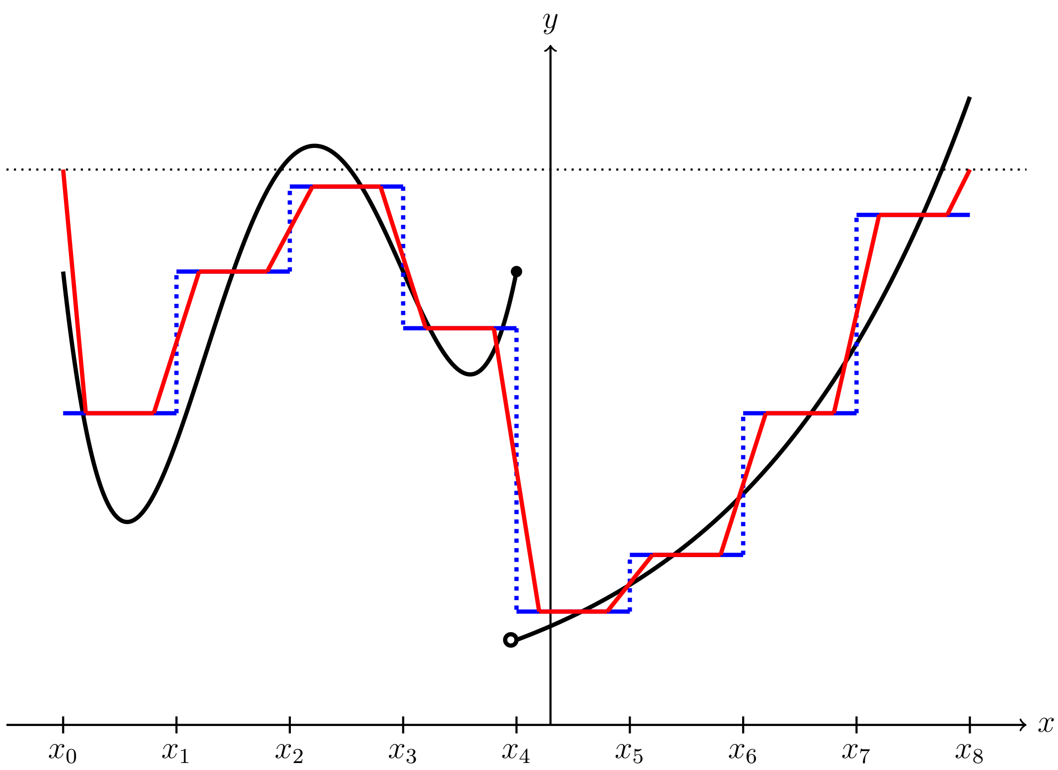
since $d_k = u(\xi_k)$ for some ξ_k such that $x_k < \xi_k \le x_{k+1}$. The inequality might not hold at the end-points, but this does not matter for the integral. This implies that

$$\int_{x_i}^{x_{i+1}} |u(x) - h(x)|^2 \, dx \le \frac{\epsilon^2}{9}|x_{i+1} - x_i|, \quad i = 0, 1, 2, \dots, n-1,$$

so

$$\|u - h\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x) - h(x)|^2 \, dx = \frac{1}{2\pi}\left(\sum_{k=0}^{n-1}\int_{x_i}^{x_{i+1}} |u(x) - h(x)|^2 \, dx\right)$$
$$\le \frac{1}{2\pi}\left(\sum_{k=0}^{n-1}\frac{\epsilon^2}{9}|x_{i+1} - x_i|\right) = \frac{\epsilon^2}{9}.$$

Next step is to approximate h by a continuous function f such that $f(-\pi) = f(\pi)$. To this end, choose a $\delta > 0$ such that $\delta < \dfrac{\pi\epsilon^2}{36M^2 n}$ (yeah.. we'll get to that), where M is some number such that $|h(x)| \le M$ for all x. Define f such that $f(x) = d_k$ when $x_k + \delta \le x \le x_{k+1} - \delta$ and between these intervals, a straight line that connects the y-values d_k with d_{k+1}. At the endpoints, we connect d_0 and d_n with the y-value that is the mean value of $u(-\pi)$ and $u(\pi)$. See the red graph below.



The function f is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. We extend f periodically to **R**.

Since $f = h$ on large chunks of $[-\pi, \pi]$, we now note that

$$\begin{aligned}
\|f-h\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-h(x)|^2\,dx \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi+\delta} |f(x)-h(x)|^2\,dx + \sum_{k=1}^{n-1}\int_{x_k-\delta}^{x_k+\delta} |f(x)-h(x)|^2\,dx + \int_{\pi-\delta}^{\pi} |f(x)-h(x)|^2\,dx\right) \\
&\le \frac{1}{2\pi}\left(4M^2(\delta+(n-1)\cdot 2\delta+\delta)\right) = \frac{4M^2 n\delta}{\pi} \\
&\le \frac{\epsilon^2}{9},
\end{aligned}$$

where we used the rough estimate $|f(x)-h(x)| \le 2M$ which holds if $|f(x)| \le M$ (which implies that $|h(x)| \le M$ as well). Note that $f' \in E$.

5.4 Parseval's Formula

Recall from Lecture 2 that Parsevals's identity holds for closed ON systems (and we just proved this for E):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx, \quad k \in \mathbf{Z}.$$

Furthermore, this could be generalized as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx = \sum_{k=-\infty}^{\infty} c_k\overline{d_k},$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx \quad \text{and} \quad d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x)e^{-ikx}\,dx, \quad k \in \mathbf{Z}.$$

Example

Calculate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$.

Solution. Note that $u(x) = x$, $-\pi \le x < \pi$, has the Fourier coefficients $c_k = i(-1)^k/k$ for $k \neq 0$ and $c_0 = 0$ (show this). Hence

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = 2\sum_{k=1}^{\infty}\frac{1}{k^2}$$

and by Parseval's identity this series is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3},$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

This is one way of proving this famous formula.

Chapter 6

The Fourier Transform

"Crom! Grant me revenge. And if you're not listening, to hell with you!"
—Conan

6.1 The Fourier Transform

Formally, we can consider the **Fourier transform** of a function $u\colon \mathbf{R} \to \mathbf{C}$ given by

$$\mathcal{F}\, u(\omega) = \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx, \quad \omega \in \mathbf{R},$$

when this integral exists. When is this the case? Well, if $u \in L^1(\mathbf{R})$ then this integral will be absolutely integrable since $|u(x)e^{-i\omega x}| \leq |u(x)|$ (for ω real) so

$$|\,\mathcal{F}\, u(\omega)| \leq \int_{-\infty}^{\infty} |u(x)|\, dx < \infty.$$

Note that this bound is uniform in ω, so we have actually proved that

$$\|\,\mathcal{F}\, u\|_\infty \leq \|u\|_{L^1(\mathbf{R})},$$

meaning that the Fourier transform maps functions from $L^1(\mathbf{R})$ into $L^\infty(\mathbf{R})$. This space will be to hard for us to handle properly though, so let's consider piecewise continuous functions similarly with how we handled Fourier series. In some cases you'll see that $\omega = 2\pi f$ is used. This is to obtain results in terms of frequency (not angular frequency) with the unit Hertz. This won't happen very often in this course, but is quite common in signal processing.

The space $G(\mathbf{R})$

Definition. We define the space $G(\mathbf{R})$ (or just G if the domain is clear from the context) to consist of all piecewise continuous functions $u\colon \mathbf{R} \to \mathbf{C}$ that are absolutely integrable. A function is called piecewise continuous on $\mathbf{R}$ if there is a finite number of exception points in each finite interval $[a, b]$ (meaning that $u \in E[a, b]$).

Note that this means that a function in G might have an infinite number of discontinuity points (but still countably many). The simplest example is probably the *integer function* $u(x) = [x]$ that maps a real value x to its integer part.

When dealing with Fourier transforms, there's some slight variations in the notation. The most common ways to denote the Fourier transform of $u\colon \mathbf{R} \to \mathbf{C}$ are

$$U(\omega) = \widehat{u}(\omega) = \mathcal{F}\, u(\omega).$$

Choose which one you prefer and try to stay consistent (I probably won't..). Note that there are certain instances where a certain notation makes things easier to read, so some variation is alright.

When using $\mathcal{F}\, u(\omega)$, observe that this means that *the function* $\mathcal{F}\, u$ has the argument ω. If we wish to be very careful, we sometimes write $\mathcal{F}(u(x))(\omega)$ to indicate that u is a function of x and the Fourier transform of u is a function of ω, even if this notation is slightly incorrect (u is the function and $u(x)$ is the functions value at x). We might even write $(\mathcal{F}(u(x)))(\omega)$ if it helps make something clear, but clumsier notation is most often not the best idea.

Normalizing constants

There are several, different competing "versions" of the Fourier transform that differs by a constant. In the book the Fourier transform is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx$$

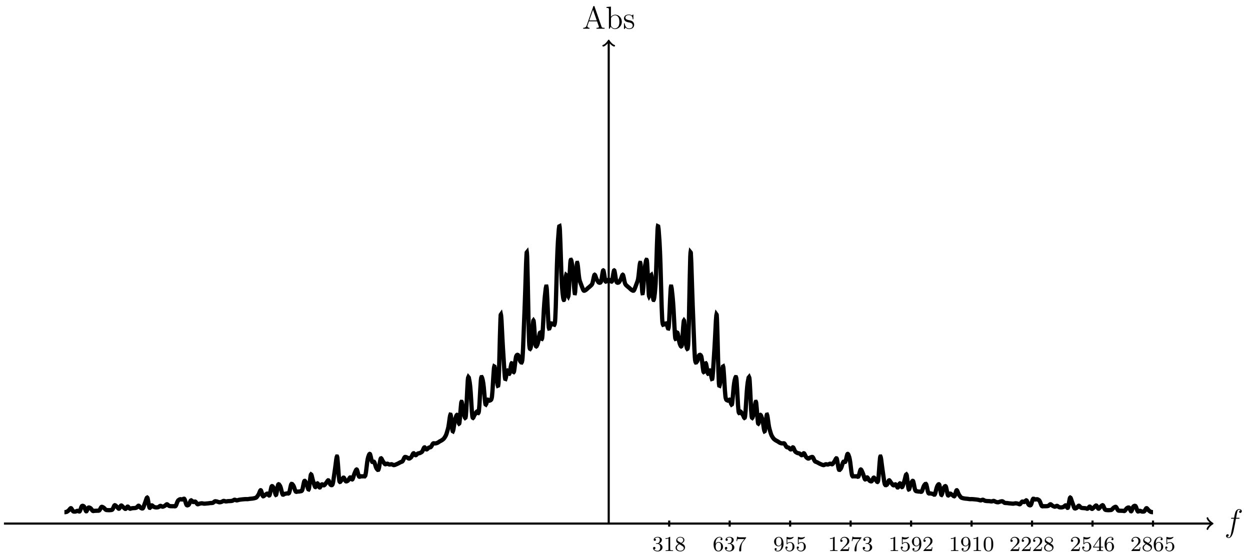
and in other material you might find that the Fourier transform is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx.$$

The theory will look the same, but obviously the Fourier transforms of specific functions will have different constants attached. Be *very* careful when reading tables! This problem will also return next lecture when we discuss the inverse Fourier transform.

6.2 Time/Space and Frequency; The Spectrum

We often think of the function $u\colon \mathbf{R} \to \mathbf{C}$ as a function of time or space, meaning that we have values at certain times or at certain points. Taking the Fourier transform of u produces a function $U\colon \mathbf{R} \to \mathbf{C}$, and we consider $U(\omega)$ as a function of angular frequency ω. If we plot the magnitude of U (that is we plot the absolute value), we typically obtain something like this.



Why this example? Why these numbers? Why does the graph look symmetric around the y-axis? So many questions. The connection between the angular frequency and regular frequency is given by

$$\omega = 2\pi f,$$

where f is the regular frequency (measured in Hertz). For an audio signal, we typically consider frequencies below 22 kHz so that's the reason for those numbers. Furthermore, A real-valued function always has a symmetric spectrum. So that's the reason for the symmetry. We'll prove that later on (it's not that difficult). For this reason we usually only plot half of the magnitude spectrum in the case when the signal is real.

6.3 Examples

A lot of calculations to derive the Fourier transforms of given functions are rather difficult in that they involve techniques that aren't available to us (like residue calculus from complex analysis). Other problems arrive from our choice of *domain* for the Fourier transform, that is, the space $G(\mathbf{R})$. Not only are we requiring functions to be piecewise continuous, but also absolutely integrable. For example, could we assign a Fourier transform to a non-zero constant? We could, but that basically requires *distribution theory* (and the answer is basically the *Dirac "function"*). So what this means is that we're going to see tables where Fourier transforms are listed that might not be completely in line with what we're able to prove, but we will use these anyway if needed. Be aware though that we have not covered the necessary theory in that case. So let's consider some examples we actually can derive without any issues.

Example

Show that the Fourier transform of $u(x) = e^{-|x|}$, $x \in \mathbf{R}$, is given by $U(\omega) = \dfrac{2}{1+\omega^2}$.

Solution. We note that $u \in G(\mathbf{R})$ and

$$\int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}\,dx = \int_{-\infty}^{0} e^{x(1-i\omega)}\,dx + \int_{0}^{\infty} e^{-x(1+i\omega)}\,dx = \left[\frac{e^{x(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0} + \left[-\frac{e^{-x(1+i\omega)}}{1+i\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{1+i\omega+(1-i\omega)}{(1+i\omega)(1-i\omega)} = \frac{2}{1+\omega^2}.$$

We also note the following partial result from the previous calculation.

Example

The Fourier transform of $u(x) = e^{-x}$, $x \geq 0$, is given by $U(\omega) = \dfrac{1}{1 + i\omega}$.

We see that situations where functions are defined from a certain point onward, the following function can be helpful in writing down such expressions.

The Heaviside Function

Definition. The **Heaviside function** H is defined by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$.

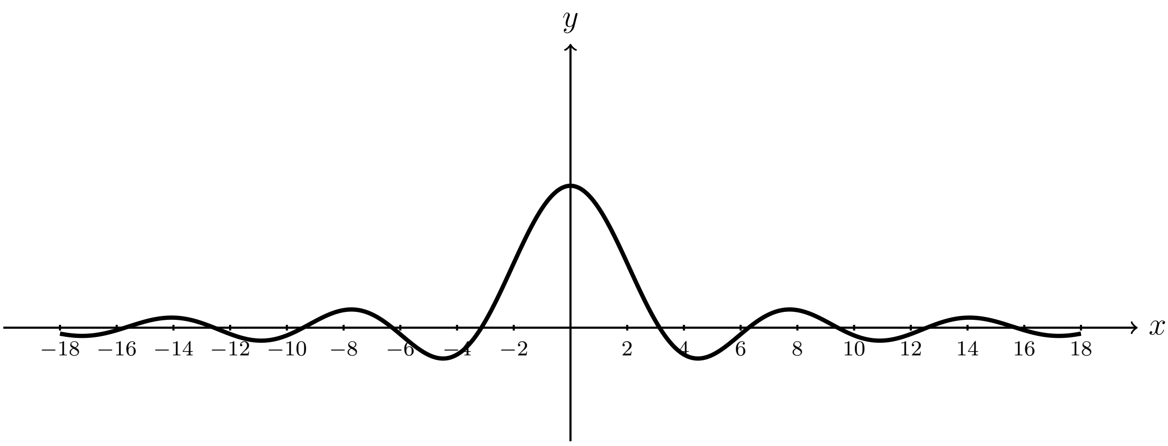
The sinc function

Definition. We define the **sinc**-function by

$$\operatorname{sinc}(x) = \frac{\sin x}{x}, \quad x \neq 0,$$

and $\operatorname{sinc}(0) = 1$ (why?).

The sinc-function is a sinusoid that decays as $1/x$. As we shall see, it is also an important function when dealing with Fourier transforms.



Example

Show that the Fourier transform of $u(x) = 1$, $x \in [-1, 1]$, and $u(x) = 0$ elsewhere, is given by $U(\omega) = 2 \operatorname{sinc} \omega$.

Solution. We note that $u \in G(\mathbf{R})$ and that

$$\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = \int_{-1}^{1} e^{-i\omega x}\,dx = \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{1} = \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = 2\,\text{sinc}\,\omega, \quad \omega \neq 0.$$

For $\omega = 0$, we find that $\mathcal{F}\,u(0) = 2$. This is $2\,\text{sinc}(0)$ so the Fourier transform of the "box" is continuous also at the origin. We will show that the continuity of the Fourier transform is true for any $u \in G(\mathbf{R})$.
Note also the contrast between the graphs of the function and its Fourier transform. Indeed, the Fourier transform (while decaying) is oscillating around the ω-axis all the way to infinity, whereas the function u is extremely limited with respect to x (it's equal to zero outside $[-1, 1]$). This is an intrinsic property of the Fourier transform. We can't have something that's both limited in x and ω at the same time. You're going to see this phenomenon in a lot of applied settings ranging from quantum mechanics (hello Heisenberg) to telecommunication.

6.4 Properties of the Fourier Transform

In the previous examples, we saw that a real valued function might give both real and complex valued Fourier transforms, but in the case when the function was symmetric we obtained a real valued transform. Is this true in general? Or was there something else that happened in these examples that produced the result? Or was it just coincidence? These types of symmetry questions and general properties of the Fourier transform are important and also what enables us to develop useful concise tables that work together with certain rules. So let's take a look at the properties and rules of the Fourier transform.

Theorem. For $u \in G$, the Fourier transform $\mathcal{F}\,u$ is uniformly continuous on $\mathbf{R}$.

Proof. So... there's an easy way of doing this by means of the Lebesgue dominated convergence theorem. However, this is slightly outside the course, so let's try something else. Let $U(\omega)$ be the Fourier transform of $u \in G$ and let h be a small real number. Then

$$|U(\omega + h) - U(\omega)| = \left|\int_{-\infty}^{\infty} u(x)\left(e^{-i(\omega+h)x} - e^{-i\omega x}\right)\,dx\right| \leq \int_{-\infty}^{\infty} |u(x)|\left|e^{-i(\omega+h)x} - e^{-i\omega x}\right|\,dx.$$

Now, we need to do something with the difference of complex exponentials. A rough estimate is given by

$$\left|e^{-i(\omega+h)x} - e^{-i\omega x}\right| \leq \left|e^{-i(\omega+h)x}\right| + \left|e^{-i\omega x}\right| = 2$$

so at least it is bounded. However, clearly the difference also goes to zero as $h \to 0$, so we can do better. Indeed, let $\alpha, \beta \in \mathbf{R}$. Then

$$\begin{aligned}\left|e^{i\alpha} - e^{i\beta}\right|^2 &= |\cos\alpha + i\sin\alpha - \cos\beta - i\sin\beta|^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2(1 - \cos(\alpha - \beta)) = 4\sin^2\left(\frac{\alpha-\beta}{2}\right) \leq 4\left(\frac{\alpha-\beta}{2}\right)^2 = (\alpha - \beta)^2,\end{aligned}$$

since $|\sin x| \leq |x|$ for $x \in \mathbf{R}$. This implies that

$$\left|e^{-i(\omega+h)x} - e^{-i\omega x}\right| \leq |-(\omega + h)x + \omega x| = |h||x|.$$

Let $\epsilon > 0$. We will prove that there exists $\delta > 0$ such that

$$|h| < \delta \quad \Rightarrow \quad |U(\omega + h) - U(\omega)| < \epsilon \text{ for every } \omega \in \mathbf{R}. \tag{6.1}$$

Since u is absolutely integrable, there exists some number $R > 0$ such that

$$\int_{|x|>R} |u(x)|\, dx < \frac{\epsilon}{4}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)| \left|e^{-i(\omega+h)x} - e^{-i\omega x}\right| dx &= \int_{|x|>R} |u(x)| \left|e^{-i(\omega+h)x} - e^{-i\omega x}\right| dx \\ &\quad + \int_{-R}^{R} |u(x)| \left|e^{-i(\omega+h)x} - e^{-i\omega x}\right| dx \\ &\leq 2 \int_{|x|>R} |u(x)|\, dx + \int_{-R}^{R} |u(x)||h||x|\, dx \\ &\leq 2\frac{\epsilon}{4} + |h|R \int_{-R}^{R} |u(x)|\, dx \leq \frac{\epsilon}{2} + |h|R \int_{-\infty}^{\infty} |u(x)|\, dx,\end{aligned}$$

so we can choose $\delta = \dfrac{1}{2R\|u\|_{L^1}}$ to obtain (6.1). □

The Riemann-Lebesgue "Lemma"

Theorem. For $u \in G$ we have $\mathcal{F}\, u(\omega) \to 0$ as $|w| \to \infty$.

Proof. Let $\epsilon > 0$. We prove that there exists $N > 0$ such that

$$|\omega| > N \quad \Rightarrow \quad |U(\omega)| < \epsilon. \tag{6.2}$$

Since u is absolutely integrable, there exists $M > 0$ such that

$$\int_{|x|>M} |u(x)|\, dx < \frac{\epsilon}{3}. \tag{6.3}$$

Since u is Riemann integrable on $[-M, M]$, there exists a step function h such that

$$\int_{-M}^{M} |u(x) - h(x)|\, dx < \frac{\epsilon}{3} \tag{6.4}$$

(we could take the lower sum for instance so that $|u(x) - h(x)| = u(x) - h(x)$). Let the step function be equal to the constant c_k when $x_k < x < x_{k+1}$, $k = 0, 1, \ldots, m-1$, where

$$-M = x_0 < x_1 < x_2 < \cdots < x_m = M$$

is a suitable partition of $[-M, M]$. Observe now that for $\omega \neq 0$,

$$\begin{aligned}\left|\int_{x_k}^{x_{k+1}} h(x) e^{-i\omega x}\, dx\right| &= \left|\int_{x_k}^{x_{k+1}} c_k e^{-i\omega x}\, dx\right| = \left| c_k \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{x=x_k}^{x=x_{k+1}} \right| \leq \frac{|c_k|}{|\omega|} \left|e^{-i\omega x_{k+1}} - e^{-i\omega x_k}\right| \\ &\leq \frac{2|c_k|}{|\omega|}.\end{aligned}$$

Hence

$$\left|\int_{-M}^{M} h(x)e^{-i\omega x}\,dx\right| \le \sum_{k=0}^{m-1}\left|\int_{x_k}^{x_{k+1}} h(x)e^{-i\omega x}\,dx\right| \le \frac{2}{|\omega|}\sum_{k=0}^{m-1}|c_k|.$$

Let $N > 6\epsilon^{-1}\sum_{k=0}^{m-1}|c_k|$. Then, if $|w| > N$, we have

$$\left|\int_{-M}^{M} h(x)e^{-i\omega x}\,dx\right| < \frac{\epsilon}{3}. \tag{6.5}$$

By equations (6.3), (6.4) and (6.5), we obtain

$$\begin{aligned}|U(\omega)| &= \left|\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx\right| \\ &\le \int_{|x|>M}|u(x)|\,dx + \left|\int_{-M}^{M}(u(x)-h(x))e^{-i\omega x}\,dx\right| + \left|\int_{-M}^{M} h(x)e^{-i\omega x}\,dx\right| \\ &< \frac{\epsilon}{3} + \int_{-M}^{M}|u(x)-h(x)|\,dx + \frac{\epsilon}{3} < \epsilon,\end{aligned}$$

which is (6.2). □

This result also implies the following useful result.

Corollary. If $u \in E[a,b]$ (piecewise continuous on $[a,b]$ and integrable), then

$$\lim_{M\to\pm\infty}\int_a^b u(x)\sin(Mx) = 0 \quad \text{and} \quad \lim_{M\to\pm\infty}\int_a^b u(x)\cos(Mx) = 0.$$

6.5 Rules for the Fourier Transform

Suppose throughout that $u, v \in G(\mathbf{R})$. Additional assumptions will be stated in the theorems.

Linearity

Theorem. If a, b are constants, then $\mathcal{F}(au + bv) = a\,\mathcal{F}\,u + b\,\mathcal{F}\,v$.

Proof. This follows from the linearity of the integral defining the Fourier transform.

Scaling

Theorem. If $a \neq 0$, then $\mathcal{F}(u(ax))(\omega) = \dfrac{1}{|a|}\,\mathcal{F}(u(x))\left(\dfrac{\omega}{a}\right)$.

Proof. First, assume that $a > 0$. Observing that

$$\begin{aligned}\mathcal{F}(u(ax))(\omega) &= \int_{-\infty}^{\infty} u(ax)e^{-i\omega x}\,dx = /\ y = ax\ / = \int_{-\infty}^{\infty} u(y)e^{-i\omega y/a}\,\frac{dy}{a} \\ &= \frac{1}{a}\int_{-\infty}^{\infty} u(y)e^{-i(\omega/a)y}\,dy = \frac{1}{a}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right).\end{aligned}$$

If $a < 0$, then we need to note that when doing the substitution, the limits will exchange places (so the integral goes from $+\infty$ to $-\infty$). Changing this back changes the sign of the integral, so we obtain that

$$\mathcal{F}(u(ax))(\omega) = -\frac{1}{a}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right) = \frac{1}{|a|}\,\mathcal{F}\,u\left(\frac{\omega}{a}\right).$$

Note the corollary we obtain when $a = -1$.

Sign change

Corollary. $\mathcal{F}(u(-x))(\omega) = \mathcal{F}(u(x))(-\omega)$.

However, note also the following property.

Real symmetry

Theorem. If u is real-valued, then $\mathcal{F}\,u(-\omega) = \overline{\mathcal{F}\,u(\omega)}$.

Proof. Since $u(x) \in \mathbf{R}$, we have

$$\begin{aligned}\mathcal{F}\,u(-\omega) &= \int_{-\infty}^{\infty} u(x)e^{-i(-\omega)x}\,dx = \int_{-\infty}^{\infty} u(x)e^{i\omega x}\,dx = \int_{-\infty}^{\infty} \overline{u(x)e^{-i\omega x}}\,dx \\ &= \overline{\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx} = \overline{\mathcal{F}\,u(\omega)}.\end{aligned}$$

Note that this implies that if u is real valued and $U(\omega) = \mathcal{F}\,u(\omega)$, then

$$|U(-\omega)| = |U(\omega)|, \qquad \operatorname{Re} U(-\omega) = \operatorname{Re}(\omega), \quad \text{and} \quad \operatorname{Im} U(-\omega) = -\operatorname{Im} U(\omega).$$

This means that there's symmetry around the imaginary axis for the spectrum of u.

Translation

Theorem. Suppose that $a \in \mathbf{R}$ is constant. Then $\mathcal{F}(u(x-a))(\omega) = e^{-i\omega a}(\mathcal{F}(u(x)))(\omega)$.

Proof. A simple substitution shows that

$$\begin{aligned}\mathcal{F}(u(x-a))(\omega) &= \int_{-\infty}^{\infty} u(x-a)e^{-i\omega x}\,dx = /\ y = x-a\ / = \int_{-\infty}^{\infty} u(y)e^{-i\omega(y+a)}\,dy \\ &= e^{-i\omega a}\int_{-\infty}^{\infty} u(y)e^{-i\omega y}\,dy = e^{-i\omega a}\,\mathcal{F}\,u(\omega).\end{aligned}$$

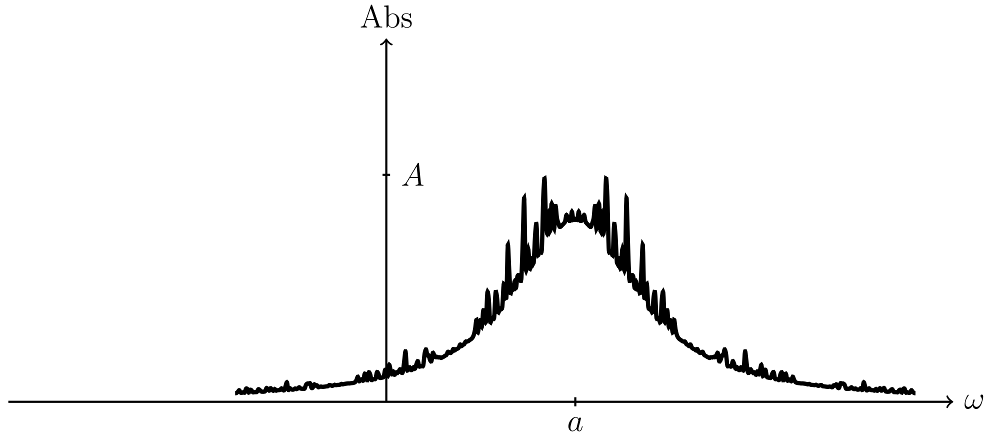
Phase shift

Theorem. Suppose that $a \in \mathbf{R}$ is constant. Then $\mathcal{F}(e^{iax}u(x))(\omega) = (\mathcal{F}(u(x)))(\omega - a)$.

Proof. We note that

$$\mathcal{F}(e^{iax}u(x))(\omega) = \int_{-\infty}^{\infty} u(x)e^{iax}e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} u(x)e^{-i(\omega-a)x}\,dx = \mathcal{F}\,u(\omega - a),$$

which completes the proof.



Euler's formulas implies the following variation (that's useful in telecommunication).

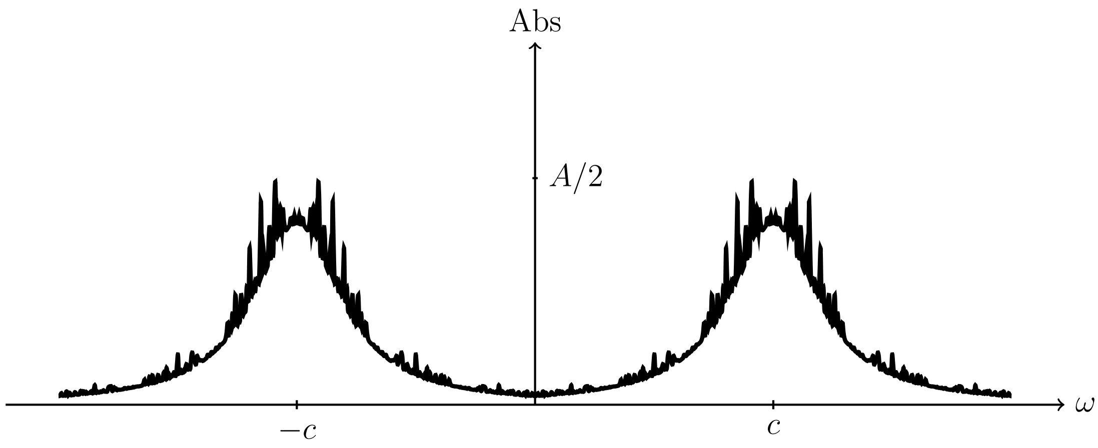
Modulation

Theorem. Suppose that $c \in \mathbf{R}$ is constant. Then

$$\mathcal{F}\big(u(x)\cos cx\big)(\omega) = \frac{\mathcal{F}(u(x))(\omega - c) + \mathcal{F}(u(x))(\omega + c)}{2}$$

and

$$\mathcal{F}\big(u(x)\sin cx\big)(\omega) = \frac{\mathcal{F}(u(x))(\omega - c) - \mathcal{F}(u(x))(\omega + c)}{2i}.$$



Notice that this means that we might get an overlap which might cause distortion in applications if the shift c is too small (if we just want a "copy" of the spectrum shifted to a higher frequency).

Complex conjugation

Theorem.

$$\mathcal{F}(\overline{u(x)})(\omega) = \overline{\mathcal{F}(u(x))(-\omega)}$$

Proof. Clearly

$$\mathcal{F}(\overline{u(x)})(\omega) = \int_{-\infty}^{\infty} \overline{u(x)} e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} \overline{u(x) e^{-i(-\omega)x}}\, dx = \overline{\int_{-\infty}^{\infty} u(x) e^{-i(-\omega)x}\, dx} = \overline{\mathcal{F}\, u(-\omega)}.$$

6.5.1 Differentiation

So let's move on to a very useful property of the Fourier transform: derivatives in one domain corresponds to multiplication by ω (or x) in the other domain. Formally, the proof is simple enough, but we need to exchange to order of integration and differentiation which is a bit problematic. So we need some preliminary results for how to handle expressions of the form $x^n u(x)$. But first, let's investigate what the Fourier transform of u' is.

Theorem. Let $u \in G(\mathbf{R})$ be differentiable and let $u' \in G(\mathbf{R})$. Then $\mathcal{F}(u')(\omega) = i\omega\, \mathcal{F}\, u(\omega)$.

Proof. First, since u is continuous, we have

$$u(x) - u(0) = \int_0^x u'(t)\, dt.$$

Since $u' \in G(\mathbf{R})$, we know that u' is absolutely integrable, and therefore the limit

$$\lim_{x\to\infty} u(x) = u(0) + \int_0^{\infty} u'(t)\, dt$$

exists. Furthermore, since u is also absolutely integrable and continuous, the limit above *must* be zero (if not then u, being continuous and having a limit at ∞, can't be absolutely integrable). Similarly we must have $u(x) \to 0$ as $x \to -\infty$. Using integration by parts, we see that

$$\int_{-M}^{M} u'(x) e^{-i\omega x}\, dx = /\ \text{I.B.P.}\ / = u(M)e^{-i\omega M} - u(-M)e^{i\omega M} + i\omega \int_{-M}^{M} u(x) e^{-i\omega x}\, dx$$
$$\to i\omega \int_{-\infty}^{\infty} u(x) e^{-i\omega x}\, dx = i\omega\, \mathcal{F}\, u(\omega), \ \text{ as } M \to \infty,$$

since $u(x) \to 0$ as $x \to \pm\infty$. □

Theorem. Let $u \in G(\mathbf{R})$ be such that $xu(x) \in G(\mathbf{R})$. Then $\mathcal{F}(xu(x))(\omega) = i\dfrac{d}{d\omega}\mathcal{F}(u(x))(\omega)$.

"Proof." Formally, the proof is rather simple. Indeed, just observing that

$$U'(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} u(x)\frac{d}{d\omega}e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} -ixu(x)e^{-i\omega x}\,dx$$
$$= -i\,\mathcal{F}(xu(x))(\omega),$$

seems to indicate that the statement is true. However, the operation of moving the differential operator inside the integral is far from trivial; see the last section of this lecture.

Example

Find the Fourier transform of the gaussian e^{-x^2}.

Solution. One way of approaching this is by observing that both e^{-x^2} and xe^{-x^2} belong to $G(\mathbf{R})$, so if $u(x) = e^{-x^2}$ and $U(\omega) = \mathcal{F}\,u(\omega)$, then

$$U'(\omega) = -i\,\mathcal{F}(xe^{-x^2})(\omega) = -i\int_{-\infty}^{\infty} xe^{-x^2}e^{-i\omega x}\,dx$$
$$= /\ \text{I.B.P.}\ / = -i\left[-\frac{1}{2}e^{-x^2}e^{-i\omega x}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\frac{-i\omega}{2}\,e^{-x^2}e^{-i\omega x}\,dx = -\frac{\omega}{2}U(\omega).$$

So U must satisfy

$$U'(\omega) + \frac{\omega}{2}U(\omega) = 0 \quad\Leftrightarrow\quad \frac{d}{d\omega}\left(e^{\omega^2/4}U(\omega)\right) = 0 \quad\Leftrightarrow\quad U(\omega) = Ce^{-\omega^2/4}.$$

However, we can only have one Fourier transform so we need to find a value for C. It is clear that

$$U(0) = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

This is a standard integral and one can for example find its value through the following calculation:

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dx\,dy$$
$$= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}\,d\theta\,dr = 2\pi\left[-\frac{1}{2}\,e^{-r^2/2}\right]_0^{\infty} = \pi.$$

Therefore $C = U(0) = \sqrt{\pi}$ and we have shown that

$$\mathcal{F}\left(e^{-x^2}\right)(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

Example

Find a solution to

$$u''(x) + 3u'(x) + 2u(x) = \begin{cases} e^{2x}, & x < 0, \\ e^{-x}, & x \geq 0. \end{cases}$$

Solution. First, note that if $a > 0$, then

$$\mathcal{F}(e^{ax}H(-x))(\omega) = \int_{-\infty}^{0} e^{ax}e^{-i\omega x}\,dx = \left[\frac{e^{(a-i\omega)x}}{a-i\omega}\right]_{-\infty}^{0} = \frac{1}{a-i\omega}.$$

Similarly, if $a > 0$, then

$$\mathcal{F}(e^{-ax}H(x))(\omega) = \int_{0}^{\infty} e^{-ax}e^{-i\omega x}\,dx = \left[-\frac{e^{-(a+i\omega)x}}{a+i\omega}\right]_{0}^{\infty} = \frac{1}{a+i\omega}.$$

Since we can express the right-hand side as $e^{2x}H(-x) + e^{-x}H(x)$, where H is the Heaviside function, we obtain (assuming that $u \in G(\mathbf{R})$ and noting that the right-hand side is also in $G(\mathbf{R})$),

$$(i\omega)^2U(\omega)+3i\omega U(\omega)+2U(\omega) = \frac{1}{2-i\omega}+\frac{1}{1+i\omega} \quad\Leftrightarrow\quad ((i\omega)^2+3i\omega+2)U(\omega) = \frac{1}{2-i\omega}+\frac{1}{1+i\omega}.$$

The right-hand side is

$$\frac{1}{2-i\omega}+\frac{1}{1+i\omega} = \frac{3}{(2-i\omega)(1+i\omega)}$$

and letting $s = i\omega$, we see (from the lefthand side) that $s^2 + 3s + 2 = (s + 1)(s + 2)$, so we're looking for whatever has the transform

$$U(\omega) = \frac{3}{(1+i\omega)^2(2+i\omega)(2-i\omega)} = \big/ \text{ partial fractions } \big/ = \frac{-2/3}{1+i\omega}+\frac{1}{(1+i\omega)^2}+\frac{3/4}{2+i\omega}+\frac{1/12}{2-i\omega}.$$

From a table (or the calculation above) we know that

$$\mathcal{F}(e^{-ax}H(X)) = \frac{1}{a+i\omega},$$

so the first and third term yields

$$-\frac{2}{3}e^{-x}H(x) + \frac{3}{4}e^{-2x}H(x).$$

Similarly, the last term yields

$$\frac{1}{12}e^{2x}H(-x).$$

To attack the remaining term, observe that

$$\frac{d}{d\omega}\left(\frac{1}{1+i\omega}\right) = -i\frac{1}{(1+i\omega)^2},$$

so since $\mathcal{F}(xu(x))(\omega) = iU'(\omega)$ (assuming that u is nice enough),

$$\mathcal{F}(xe^{-x}H(x))(\omega) = -i^2\frac{1}{(1+i\omega)^2} = \frac{1}{(1+i\omega)^2}.$$

So we see that

$$\left(x-\frac{2}{3}\right)e^{-x}H(x) + \frac{3}{4}e^{-2x}H(x) + \frac{1}{12}e^{2x}H(-x)$$

has the Fourier transform $U(\omega)$. So we suggest that

$$u(x) = \begin{cases} \dfrac{1}{12}e^{2x}, & x < 0,\\ \dfrac{3}{4}e^{-2x} + \left(x-\dfrac{2}{3}\right)e^{-x}, & x \geq 0, \end{cases}$$

is a solution to the differential equation. Directly verifying this proves the statement (something that should be done at this point). If we knew some uniqueness results, we could argue why this is the solution.

6.6 Principal Values and Integration

Definition. The **principal value** of an integral $\int_{-\infty}^{\infty} u(x)\,dx$ is defined as

$$\text{P.V.}\int_{-\infty}^{\infty} u(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} u(x)\,dx$$

whenever this limit exists.

If u is absolutely integrable, then

$$\text{P.V.}\int_{-\infty}^{\infty} u(x)\,dx = \int_{-\infty}^{\infty} u(x)\,dx.$$

In other words,

$$\lim_{R\to\infty}\int_{-R}^{R} u(x)\,dx = \lim_{m,M\to\infty}\int_{-m}^{M} u(x)\,dx$$

is finite if absolutely convergent (this was the definition in TATA42). This is clear since

$$\begin{aligned}\left|\int_{-\infty}^{\infty} u(x)\,dx - \int_{-R}^{R} u(x)\,dx\right| &= \left|\int_{-\infty}^{-R} u(x)\,dx + \int_{R}^{\infty} u(x)\,dx\right| \\ &\leq \int_{-\infty}^{-R} |u(x)|\,dx + \int_{R}^{\infty} |u(x)|\,dx \to 0,\end{aligned}$$

as $R\to\infty$ since u is absolutely integrable, which implies that both integrals in the righthand side tend to zero (independently of each other).
Note that in the case that the integrals are absolutely integrable, the principal value integral will be equal to the integral with separate limits towards the infinities. So for us, they will produce the same value.
Now, let $f\colon \mathbf{R}^2 \to \mathbf{C}$ be a function of two real variables.

Uniform convergence of a principal value integral

Definition. We say that $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on I if the integral exists for every x and

$$\sup_{x\in I}\left|\int_{-R}^{R} f(x,y)\,dy - F(x)\right| \to 0, \text{ as } R\to\infty.$$

Another useful concept (that's true in a setting a lot more general than ours) is that of dominated convergence. In a sense this is a uniform convergence, and as the following theorem shows we can use this to obtain uniform convergence as defined above.

Dominated convergence

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ and that $F(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ exists for all x. If there exists an absolutely integrable function $g\colon \mathbf{R} \to \mathbf{R}$ such that $|f(x,y)| \leq g(y)$ for all $x, y \in \mathbf{R}$, then $\int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on $\mathbf{R}$.

Proof. Let $F_R(x) = \int_{-R}^{R} f(x,y)\,dy$, $R > 0$. Since $F(x)$ exists for every x, it is clear that

$$|F(x) - F_R(x)| = \left| \int_{-\infty}^{-R} f(x,y)\,dy + \int_{R}^{\infty} f(x,y)\,dy \right| \leq \int_{-\infty}^{-R} |f(x,y)|\,dy + \int_{R}^{\infty} |f(x,y)|\,dy.$$

Observe now that $|f(x,y)| \leq g(y)$ implies that

$$\int_{-\infty}^{-R} |f(x,y)|\,dy \leq \int_{-\infty}^{-R} g(y)\,dy \to 0,$$

as $R \to \infty$ independently of x (since we know that g is absolutely integrable). Obviously the analogous result holds for $\int_{R}^{\infty} |f(x,y)|\,dy$. This proves that

$$\sup_{x\in\mathbf{R}} |F(x) - F_R(x)| \leq \int_{-\infty}^{-R} g(y)\,dy + \int_{R}^{\infty} g(y)\,dy \to 0,$$

as $R \to \infty$, which is uniform convergence. □

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,R]$. Then

(i) $F_R(x) = \int_{a}^{R} f(x,y)\,dy$ is continuous on $[c,d]$

(ii) and if in addition f is continuous on $[c,d] \times [a,\infty[$ and $F(x) = \int_{a}^{\infty} f(x,y)\,dy$ converges uniformly (on $[c,d]$), then F is continuous.

Proof. This result is dependent on the uniform continuity of f on the closed set $[c,d] \times [a,R]$ (a continuous function on a compact set is always uniformly continuous), meaning that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|(x,y) - (x_0,y_0)| < \delta \quad \Rightarrow \quad |f(x,y) - f(x_0,y_0)| < \epsilon.$$

Note that δ is independent of the points x, y, x_0, y_0 (this is the uniformity).

(i) So, let $\epsilon > 0$ be fixed and choose $\delta > 0$ such that $|f(x+h,y) - f(x,y)| < \dfrac{\epsilon}{R-a}$ when $|h| < \delta$. Then

$$\begin{aligned} |F_R(x+h) - F_R(x)| &= \left| \int_{a}^{R} \big(f(x+h,y) - f(x,y)\big)\,dy \right| \\ &\leq \int_{a}^{R} |f(x+h,y) - f(x,y)|\,dy < \frac{\epsilon}{R-a} \int_{a}^{R} dy = \epsilon, \end{aligned}$$

which proves that F_R is continuous.

(ii) Since F_R is continuous and $F_R \to F$ uniformly on the interval $[c,d]$, it follows that F is continuous on $[c,d]$. □

Exchanging the order of integration (Fubini's Theorem)

Theorem. Suppose that $f\colon \mathbf{R}^2 \to \mathbf{C}$ is a continuous function on $[c,d] \times [a,\infty[$ and that $F(x) = \int_a^\infty f(x,y)\,dy$ converges uniformly (on $[c,d]$). Then

$$\int_c^d \left(\int_a^\infty f(x,y)\,dy \right) dx = \int_a^\infty \left(\int_c^d f(x,y)\,dx \right) dy. \tag{6.6}$$

Proof. From standard multivariate analysis, we know that

$$\int_c^d \left(\int_a^R f(x,y)\,dy \right) dx = \int_a^R \left(\int_c^d f(x,y)\,dx \right) dy$$

for any constant $R > 0$. Now, by the uniform convergence, it is clear that

$$\begin{aligned}\int_a^\infty \left(\int_c^d f(x,y)\,dx \right) dy &= \lim_{R\to\infty} \int_a^R \left(\int_c^d f(x,y)\,dx \right) dy = \lim_{R\to\infty} \int_c^d \left(\int_a^R f(x,y)\,dy \right) dx \\ &= \lim_{R\to\infty} \int_c^d F_R(x)\,dx = \int_c^d \lim_{R\to\infty} F_R(x)\,dx = \int_c^d F(x)\,dx,\end{aligned}$$

which implies that (6.6) holds. □

Note that we can let $a = -\infty$ in the previous theorems by exchanging $[a,R]$ by $[-R,R]$ and consider the principal values.

Leibniz rule

Theorem. Let $f\colon \mathbf{R}^2 \to \mathbf{C}$ be continuous and let $f'_x(x,y)$ exist and also be continuous. Suppose that $\int_{-\infty}^\infty f(x,y)\,dy$ is convergent for every x and suppose that $\int_{-\infty}^\infty f'_x(x,y)\,dy$ is uniformly convergent. Then

$$F'(x) = \frac{d}{dx} \int_{-\infty}^\infty f(x,y)\,dy = \int_{-\infty}^\infty f'_x(x,y)\,dy.$$

Proof. Let $G(x) = \int_a^\infty f'_x(x,y)\,dy$. Since this integral is assumed to be uniformly convergent and f'_x is continuous, it is clear that also G is continuous. Hence, for any $b \in \mathbf{R}$,

$$\begin{aligned}\int_b^x G(t)\,dt &= \int_b^x \int_{-\infty}^\infty f'_t(t,y)\,dy\,dt = \int_{-\infty}^\infty \int_b^x f'_t(t,y)\,dt\,dy \\ &= \int_{-\infty}^\infty \big(f(x,y) - f(b,y)\big)\,dy = F(x) - F(b).\end{aligned}$$

The fact that G is continuous proves that

$$\frac{d}{dx} \int_b^x G(t)\,dt = G(x),$$

so

$$F'(x) = \frac{d}{dx}\big(F(x) - F(b)\big) = G(x) = \int_{-\infty}^{\infty} f_x'(x,y)\,dy,$$

which is precisely want we wanted to show. □

6.7 Proof that $\mathcal{F}(xu(x))(\omega) = i(\mathcal{F}\,u(\omega))'$

The assumption was that $u \in G(\mathbf{R})$ and that $xu(x)$ is absolutely integrable (well.. we assumed that this product also belonged to $G(\mathbf{R})$ but given that $u \in G(\mathbf{R})$ this is equivalent). First, let us assume that u is continuous. Since $|e^{i\omega x}| = 1$ for $\omega \in \mathbf{R}$, it follows that the integral $\mathcal{F}(xu(x))(\omega)$ converges uniformly. By Leibniz' theorem, we can thus move the differentiation inside the integral obtaining that

$$\frac{d}{d\omega}\,\mathcal{F}(u)(\omega) = \int_{-\infty}^{\infty} u(x)\frac{d}{d\omega}e^{-i\omega x}\,dx = -i\int_{-\infty}^{\infty} xu(x)e^{-i\omega x}\,dx = -i\,\mathcal{F}(xu(x))(\omega),$$

which proves the claim in the case when u is continuous. If u has points of discontinuity, say $\{a_n\}_{n\in\mathbf{Z}}$ in increasing order, then the series

$$\mathcal{F}(xu(x))(\omega) = \sum_{n\in\mathbf{Z}}\int_{a_n}^{a_{n+1}} xu(x)e^{-i\omega x}\,dx$$

will converge uniformly, so by the argument above,

$$\frac{d}{d\omega}\,\mathcal{F}(u)(\omega) = \sum_{n\in\mathbf{Z}}\frac{d}{d\omega}\int_{a_n}^{a_{n+1}} u(x)e^{-i\omega x}\,dx = -i\sum_{n\in\mathbf{Z}}\int_{a_n}^{a_{n+1}} xu(x)e^{-i\omega x}\,dx = -i\,\mathcal{F}(xu(x))(\omega).$$

Note that $xu(x)$ will have at most the same discontinuity points as u.

Chapter 7

Inversion, Plancherel and Convolution

"You should not drink and bake"
—Mark Kaminski

7.1 Inversion of the Fourier Transform

So suppose that we have $u \in G(\mathbf{R})$ and have calculated the Fourier transform $\mathcal{F}\,u(\omega)$. Can we from $\mathcal{F}\,u(\omega)$ recover the function we started with? Considering that the Fourier transform is constructed by the multiplication with $e^{-i\omega x}$ and then integration, what would happen if we multiplied with $e^{i\omega x}$ and integrate again? Formally,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega &= \lim_{R\to\infty}\int_{-R}^{R}\int_{-\infty}^{\infty} u(t)e^{-i\omega t}e^{i\omega x}\,dt\,d\omega = \lim_{R\to\infty}\int_{-R}^{R}\int_{-\infty}^{\infty} u(t)e^{-i\omega(x-t)}\,dt\,d\omega \\ &= \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t)\left(\int_{-R}^{R} e^{-i\omega(x-t)}\,d\omega\right)dt,\end{aligned}$$

where we changed the order of integration (this can be motivated) but we're left with something kind of weird in the inner parenthesis and we would probably like to move the limit inside the outer integral. First, let's look at the expression in the inner parenthesis:

$$\int_{-R}^{R} e^{-i\omega(x-t)}\,d\omega = \left[\frac{e^{-i\omega(x-t)}}{-i(x-t)}\right]_{\omega=-R}^{R} = -\frac{e^{-iR(x-t)}}{i(x-t)} + \frac{e^{iR(x-t)}}{i(x-t)} = \frac{2\sin(R(x-t))}{x-t}, \quad x \neq t.$$

The Dirichlet kernel (on the real line)

Definition. We define the **Dirichlet kernel** for the Fourier transform by

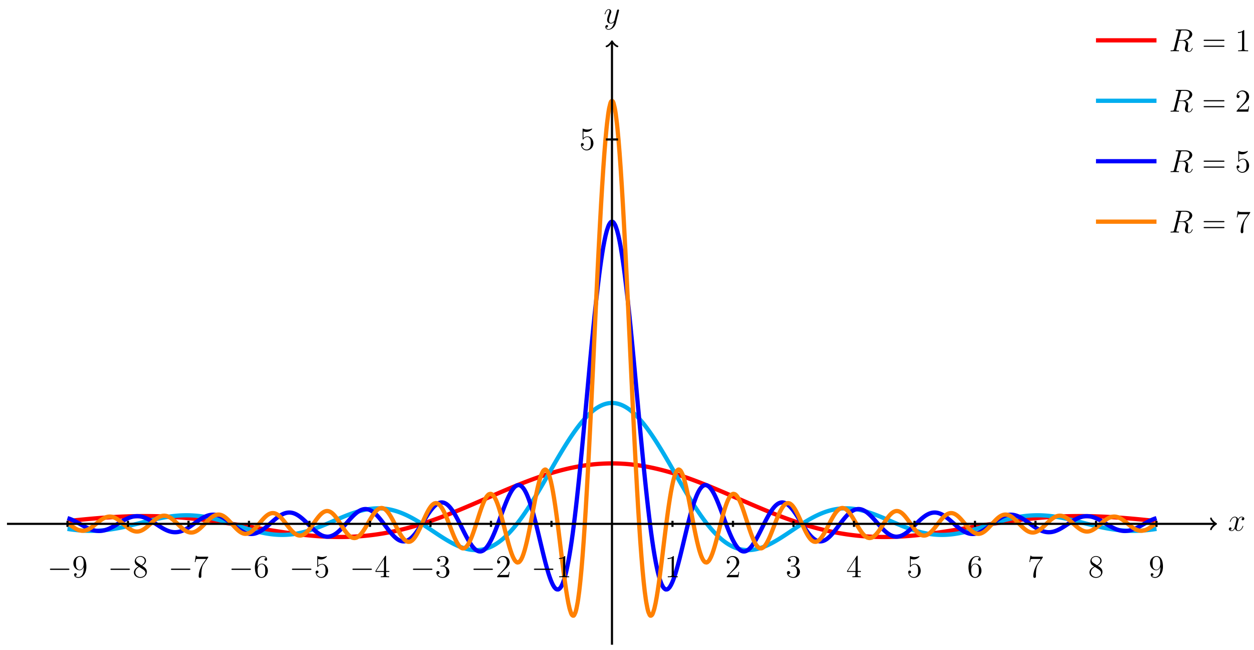
$$D_R(x) = \frac{\sin(Rx)}{\pi x}, \quad x \neq 0,\ R > 0,$$

and $D_R(0) = R/\pi$.

Note that we changed the *normalization* of the function. There's a reason for this and we'll get to that soon. For now, observe that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega = \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t)D_R(x-t)\,dt = \lim_{R\to\infty}\int_{-\infty}^{\infty} u(t+x)D_R(t)\,dt.$$

You probably recall the sinc-function, and the Dirichlet kernel on the real line is such a function and for a couple of values of R you can see the graphs below.



The Fourier inversion formula

Theorem. If $u \in G(\mathbf{R})$ has right- and lefthand derivatives at x, then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

Proof. First, we write

$$\frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) e^{i\omega x}\, d\omega = \int_{-\infty}^{0} u(t+x) D_R(t)\, dt + \int_{0}^{\infty} u(t+x) D_R(t)\, dt$$

and claim that

$$\int_{-\infty}^{0} u(t+x) D_R(t)\, dt \to \frac{u(x^-)}{2} \quad \text{and} \quad \int_{0}^{\infty} u(t+x) D_R(t)\, dt \to \frac{u(x^+)}{2},$$

as $R \to \infty$. We prove the second identity (the first is proved analogously). To this end, we split the integral in two parts:

$$\int_{0}^{\infty} u(t+x) D_R(t)\, dt = \int_{0}^{\pi} u(t+x) D_R(t)\, dt + \int_{\pi}^{\infty} u(t+x) D_R(t)\, dt.$$

The reason for this is that we need to exploit different properties of u to prove the desired result. First, let x be fixed. Then the function $t \mapsto u(t+x)$ is in $G(\mathbf{R})$, so the Riemann Lebesgue lemma implies that

$$\lim_{R\to\infty} \int_{\pi}^{\infty} u(t+x) D_R(t)\, dt = \lim_{R\to\infty} \frac{1}{\pi} \int_{\pi}^{\infty} \frac{u(t+x)}{t} \sin(Rt)\, dt = 0.$$

Turning our attention to the first integral, we write

$$\begin{aligned}\int_0^\pi u(t+x)D_R(t)\,dt &= \int_0^\pi \big(u(t+x)-u(x^+)\big)D_R(t)\,dt + \int_0^\pi u(x^+)D_R(t)\,dt\\ &= \int_0^\pi \big(u(t+x)-u(x^+)\big)D_R(t)\,dt + u(x^+)\int_0^\pi D_R(t)\,dt.\end{aligned}$$

Since $D^+u(x)$ exists (by assumption), it is clear that the difference quotient

$$\frac{u(t+x)-u(x^+)}{t}$$

is bounded and that this expression belongs to $E([0,\pi])$. Therefore, the Riemann Lebesgue lemma (again!) implies that

$$\lim_{R\to\infty}\int_0^\pi \big(u(t+x)-u(x^+)\big)D_R(t)\,dt = \lim_{R\to\infty}\frac{1}{\pi}\int_0^\pi \frac{u(t+x)-u(x^+)}{t}\sin(Rt)\,dt = 0.$$

Finally, we observe that

$$\int_0^\pi D_R(t)\,dt = \big/\ x = Rt\ \big/ = \frac{1}{\pi R}\int_0^{R\pi}\frac{\sin x}{x/R}\,dx = \frac{1}{\pi}\int_0^{R\pi}\frac{\sin x}{x}\,dx \to \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2},\ \text{as } R\to\infty,$$

due to the following result.

Theorem. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

We defer the proof of this until at the end of the lecture.

Uniqueness

Corollary. If $u, v \in G(\mathbf{R})$ and $\mathcal{F}u(\omega) = \mathcal{F}v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for all $x \in \mathbf{R}$ where u and v are continuous and $D^{\pm}u(x)$ and $D^{\pm}v(x)$ exists.

An Airy equation

Find a (formal) expression for a nonzero solution to $u''(x) - xu(x) = 0$.

Solution. Assuming that $u \in G(\mathbf{R})$ is twice differentiable with $u', u'' \in G(\mathbf{R})$, we can take the Fourier transform and obtain that

$$\begin{aligned}(i\omega)^2U(\omega) - iU'(\omega) = 0 \quad &\Leftrightarrow \quad U'(\omega) - i\omega^2U(\omega) = 0\\ &\Leftrightarrow \quad \frac{d}{d\omega}\Big(e^{-i\omega^3/3}U(\omega)\Big) = 0 \quad \Leftrightarrow \quad U(\omega) = Ce^{i\omega^3/3},\end{aligned}$$

where C is an arbitrary constant (and we used an integrating factor to solve the differential equation). Therefore,

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ce^{i\omega^3/3}e^{i\omega x}\,d\omega = D\int_{-\infty}^{\infty} e^{i(\omega^3/3+\omega x)}\,d\omega,$$

where D is some constant, might be an expression for a solution. Now the question is of course if this integral is convergent. Certainly it is not absolutely integrable (why?) and we can't claim that the expression solves the equation by previous results. This is an instance where we would like to extend the Fourier transform to a larger class of functions.

7.2 The Fourier Transform of the Fourier Transform

So looking at the inverse Fourier transform, it's almost the same as the Fourier transform. Indeed, the only difference is the sign in the exponent of the exponential and the factor before the integral. This means that the inverse transform has pretty much the same properties as the Fourier transform. This also means the following useful result.

Theorem. If $u, U \in G(\mathbf{R})$ and $U(\omega) = \mathcal{F}(u)(\omega)$, then

$$\mathcal{F}^{-1}(U)(x) = \frac{1}{2\pi}\mathcal{F}\big((\mathcal{F}\,u)(-\omega)\big)(x) \quad \text{and} \quad \mathcal{F}(\mathcal{F}\,u(\omega))(x) = 2\pi u(-x),$$

for every x where u is continuous and $D^{\pm}u(x)$ exist.

This follows immediately from the definitions of the transforms and the result above. The assumption that $D^{\pm}u(x)$ exist is superfluous but we do not know that at this point (we'll show that next lecture). If u is discontinuous, but still in $G(\mathbf{R})$, then the equalities still hold if we view the results as elements from $L^1(\mathbf{R})$, meaning that the difference has L^1-norm zero.

Example

Find the Fourier transform of $\dfrac{1}{1+x^2}$.

Solution. Let $u = \frac{1}{2}e^{-|x|}$. We know from before that $\mathcal{F}(u) = \mathcal{F}(e^{-|x|}/2)(\omega) = \dfrac{1}{1+\omega^2}$, and since both u and $x \mapsto \dfrac{1}{1+x^2}$ belong to $G(\mathbf{R})$ and are continuous with right- and lefthand derivatives at every point, we find that

$$2\pi \cdot \frac{1}{2}e^{-|-x|} = \mathcal{F}(\mathcal{F}\,u(\omega))(x) = \mathcal{F}\left(\frac{1}{1+(-\omega)^2}\right)(x) = \mathcal{F}\left(\frac{1}{1+\omega^2}\right)(x).$$

In other words,

$$\mathcal{F}\left(\frac{1}{1+x^2}\right)(\omega) = \pi e^{-|\omega|}.$$

7.3 Convolution

A useful type of "product" of two functions is the convolution (*sv. faltning*), defined as follows.

Convolution

Definition. The convolution $u * v\colon \mathbf{R} \to \mathbf{C}$ of two functions $u\colon \mathbf{R} \to \mathbf{C}$ and $v\colon \mathbf{R} \to \mathbf{C}$ is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(t)v(x-t)\,dt, \quad x \in \mathbf{R},$$

whenever this integral exists.

So when *does* this integral exist?

Theorem. If $u, v \in G(\mathbf{R})$, then $u * v \in G(\mathbf{R})$.

Proof. We first prove that $u * v$ is absolutely integrable:

$$\begin{aligned}\int_{-\infty}^{\infty} |u * v(x)|\, dx &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} u(t)v(x-t)\, dt \right| dx \\ &\leq \text{/ monotonicity /} \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(t)v(x-t)|\, dt\, dx \\ &= \text{/ Fubini /} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(t)v(x-t)|\, dx\, dt \\ &= \int_{-\infty}^{\infty} |u(t)| \int_{-\infty}^{\infty} |v(x-t)|\, dx\, dt.\end{aligned}$$

Note now that

$$\int_{-\infty}^{\infty} |v(x-t)|\, dx = \text{/ } s = x - t \text{ /} = \int_{-\infty}^{\infty} |v(s)|\, ds,$$

so

$$\int_{-\infty}^{\infty} |u(t)| \int_{-\infty}^{\infty} |v(x-t)|\, dx\, dt = \left(\int_{-\infty}^{\infty} |u(t)|\, dt \right) \left(\int_{-\infty}^{\infty} |v(s)|\, ds \right) < \infty.$$

A more compact way of stating this result is that

$$\|u * v\|_{L^1(\mathbf{R})} \leq \|u\|_{L^1(\mathbf{R})} \|v\|_{L^1(\mathbf{R})}.$$

The right-hand side is finite by assumption. □

Note. This result holds for any integrable functions (we do *not* need the piecewise continuity to prove convergence).

7.3.1 So What *Is* the Convolution?

The convolution is a type of moving average, where we shape one function by another. There are many (seriously.. there are a lot of them) applications where convolutions appear. Linear systems, (partial) differential equations, probability theory, integration theory, etc.

Example

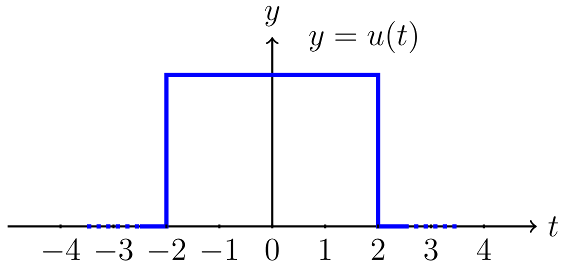
Let $u(x) = 5H(x+2) - 5H(x-2)$ and $v(x) = 4H(x+1) - 4H(x-2)$, that is,

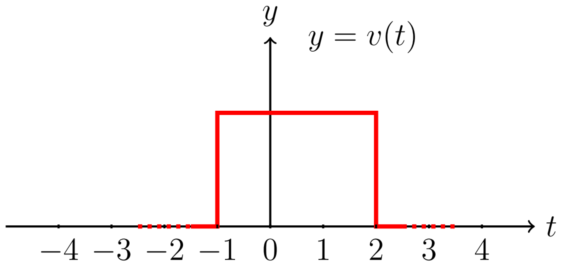
$$u(x) = \begin{cases} 5, & -2 \leq x \leq 2, \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 4, & -1 \leq x \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the convolution $u * v(x)$.

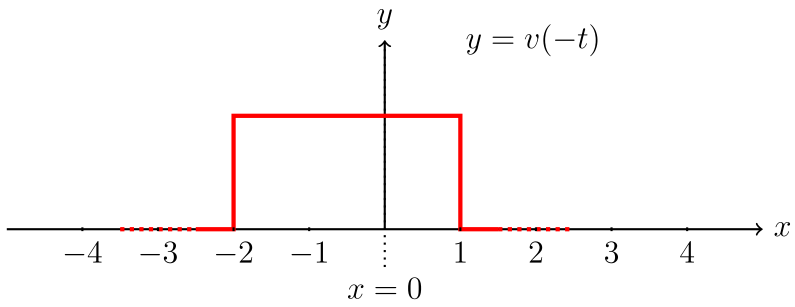
Solution. Since both functions are defined by cases, a reasonable procedure is as follows.

(i) First, identify where the functions have jumps (or where the *support*[1] is if it is compact). We also express both functions in terms of a variable t that's going to disappear when we integrate.



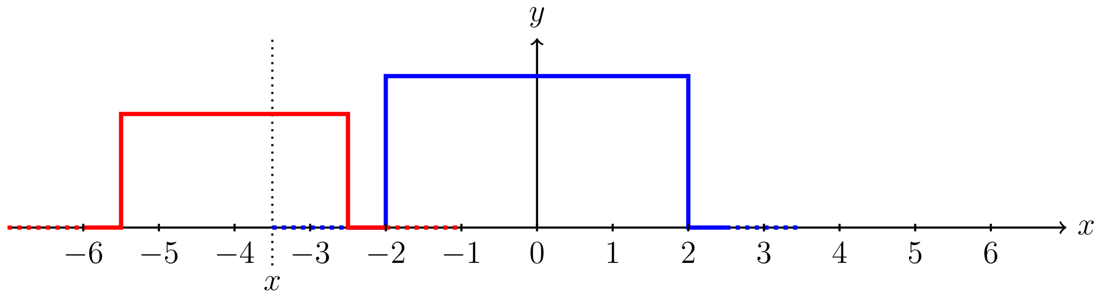


(ii) Now we mirror v, so lets draw $y = v(-t)$. Since we will consider $v(x - t)$, this graph corresponds to $x = 0$. We need to keep track of where x is.



(iii) Draw both $u(t)$ and $v(x - t)$ in the same diagram, identifying when things change.

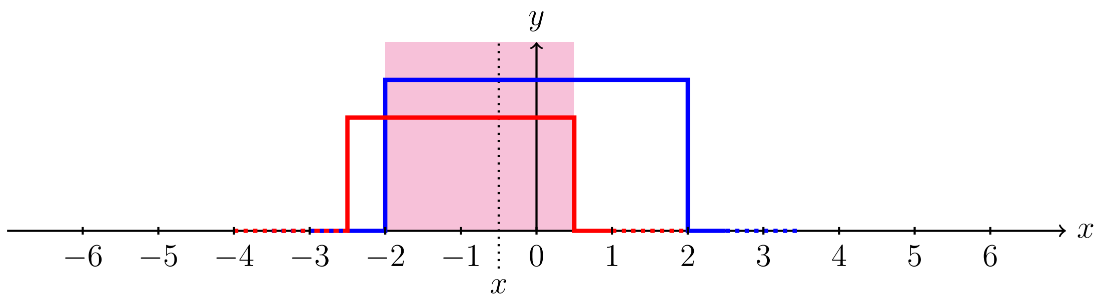
First, we see that for $x < -3$, we have no overlap.



Obviously, $u * v(x) = 0$ for $x < -3$.

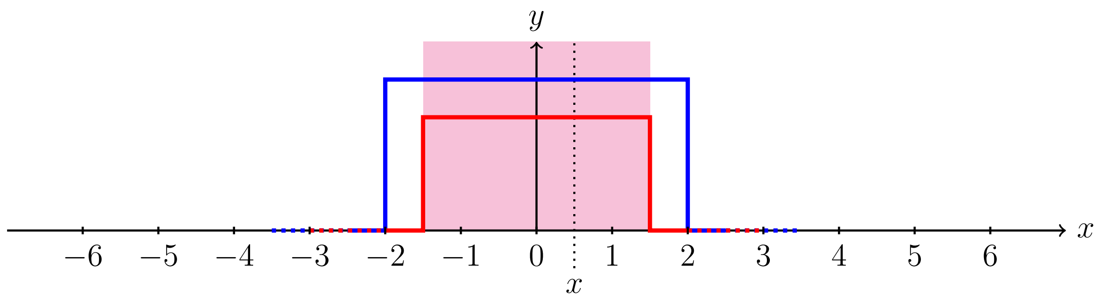
For $-3 \leq x \leq 0$, we have some overlap:

[1]The support of a function $u\colon \mathbf{R} \to \mathbf{C}$ is the smallest closed set E such that $\{x \in \mathbf{R} : u(x) \neq 0\} \subset E$.



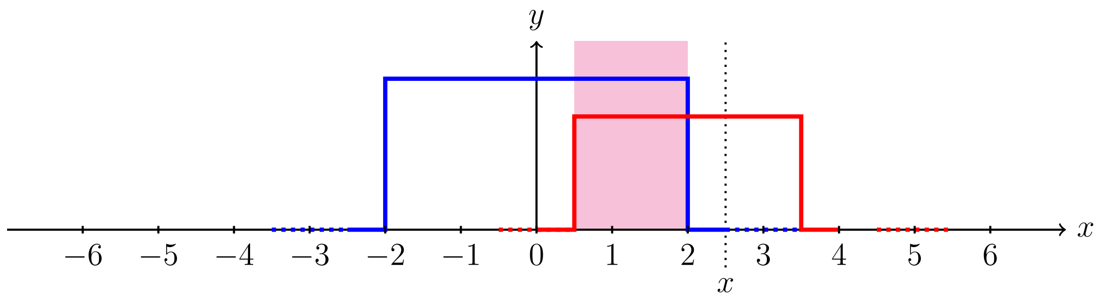
$$u * v(x) = \int_{-2}^{x+1} u(t)v(x-t)\,dt = \int_{-2}^{x+1} 5 \cdot 4\,dt = 20(x+3).$$

For $0 \leq x \leq 1$, we have complete overlap:



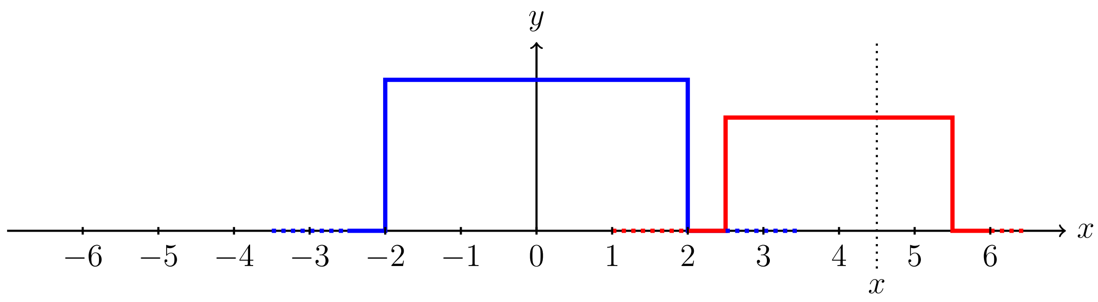
$$u * v(x) = \int_{x-2}^{x+1} u(t)v(x-t)\,dt = \int_{x-2}^{x+1} 5 \cdot 4\,dt = 20(x+1-x+2) = 20 \cdot 3.$$

For $1 \leq x \leq 4$, we have some overlap:



$$u * v(x) = \int_{x-2}^{2} u(t)v(x-t)\,dt = \int_{x-2}^{2} 5 \cdot 4\,dt = 20(4-x).$$

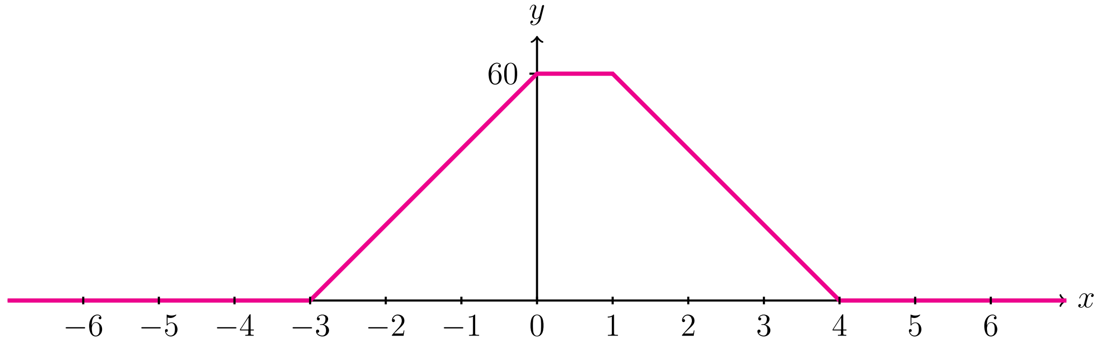
For $x > 4$, there is no overlap so $u * v(x) = 0$.



We have now covered all possibilities for x, so the answer is

$$u * v(x) = \begin{cases} 0, & x < -3, \\ 20(x+3), & -3 \le x < 0, \\ 60, & 0 \le x < 1, \\ 20(4-x), & 1 \le x \le 4, \\ 0, & x > 4, \end{cases}$$

and the graph looks like this.



7.3.2 The Fourier Transform

So now to one of the most important properties of the Fourier transform: the Fourier transform of the convolution of u and v is the product of the Fourier transforms of u and v (separately).

Convolution

Theorem. Suppose that $u, v \in G(\mathbf{R})$. Then $\mathcal{F}(u * v)(\omega) = \mathcal{F}\,u(\omega)\,\mathcal{F}\,v(\omega)$.

Proof. Let $\mathcal{F}\,u(\omega) = U(\omega)$ and $\mathcal{F}\,v(\omega) = V(\omega)$. Then

$$\begin{aligned} \mathcal{F}(u * v)(\omega) &= \int_{-\infty}^{\infty} (u * v)(x) e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(t) v(x-t)\,dt \right) e^{-i\omega x}\,dx \\ &= /\text{ Fubini } / = \int_{-\infty}^{\infty} u(t) \int_{-\infty}^{\infty} v(x-t) e^{-i\omega x}\,dx\,dt = \int_{-\infty}^{\infty} u(t) V(\omega) e^{-i\omega t} dt \\ &= V(\omega) \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt = V(\omega) U(\omega). \end{aligned}$$

Example

Find a solution to the integral equation

$$\int_{-\infty}^{\infty} u(t)u(x-t)\,dt = e^{-x^2}.$$

Solution. The left-hand side is the convolution of u with itself. Assume that $u \in G(\mathbf{R})$. Then taking the Fourier transform of both sides in the equality yields

$$U(\omega)U(\omega) = \mathcal{F}(e^{-x^2})(\omega) = e^{-\omega^2/4}$$

so assuming that U is real-valued (is this obvious?),

$$|U(\omega)| = \sqrt{e^{-\omega^2/4}} = e^{-\omega^2/8},$$

so

$$U(\omega) = \pm e^{-\omega^2/8} = \pm\sqrt{2}\cdot\frac{1}{\sqrt{2}}e^{-(\omega/\sqrt{2})^2/4} = \pm\sqrt{2}\cdot\frac{1}{\sqrt{2}}F\left(\frac{\omega}{\sqrt{2}}\right),$$

where we rewrote the right-hand side in term of $F(\omega) = \mathcal{F}(e^{-x^2})(\omega)$. Hence

$$u(x) = \pm\sqrt{2}e^{-(x\sqrt{2})^2} = \pm\sqrt{2}e^{-2x^2},$$

by a scaling argument. Is this a solution? Yes, by uniqueness (obviously u is continuously differentiable).

7.3.3 The Fourier Transform of a Product

So is there a way of finding the Fourier transform of a product? As it turns out, there is. At least if we are willing to calculate a convolution in the frequency domain (assuming things are defined).

Fourier Transform of a product

Theorem. Suppose that $u, v \in G(\mathbf{R})$ such that $uv, \mathcal{F}u, \mathcal{F}v \in G(\mathbf{R})$. Then

$$\mathcal{F}(uv)(\omega) = \frac{1}{2\pi}\mathcal{F}(u) * \mathcal{F}(v)(\omega). \tag{7.1}$$

Proof. We observe that

$$\mathcal{F}\left(\frac{1}{2\pi}(\mathcal{F}(u) * \mathcal{F}(v))(\omega)\right)(x) = \frac{1}{2\pi}\mathcal{F}(\mathcal{F}u)(x)\,\mathcal{F}(\mathcal{F}v)(x) = \frac{1}{2\pi}\cdot 2\pi u(-x)2\pi v(-x)$$
$$= 2\pi(uv)(-x) = \mathcal{F}(\mathcal{F}(uv)(\omega))(x),$$

so the Fourier transform of the left-hand and right-hand sides are equal. Assuming that all integrands belong to $G(\mathbf{R})$, the equality in (7.1) follows from a uniqueness result we show next lecture (at least except for a countable set of points). □

Note that there are a lot of things that need to align correctly for the previous result to hold. Functions and their respective transforms need to belong to $G(\mathbf{R})$, even if that assumption looks unnecessary from the final formula. Let's look at an example where this problem becomes clear.

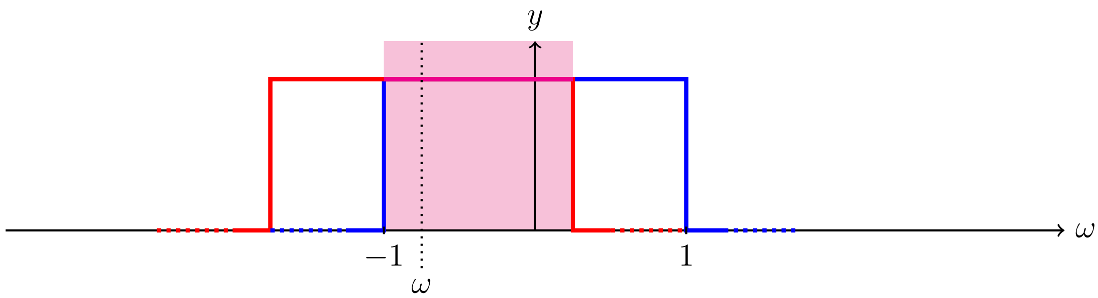
Example

Formally find the Fourier transform of $u(x) = \operatorname{sinc}(x)^2 = \left(\dfrac{\sin x}{x}\right)^2$.

Solution. Recall that the Fourier transform of $v(x) = 1$ when $-1 \le x \le 1$ and $v(x) = 0$ elsewhere, was $\mathcal{F}\, v(\omega) = 2\operatorname{sinc}(\omega)$. This would indicate that $\mathcal{F}(2\operatorname{sinc}(\omega))(x) = 2\pi v(-x)$, which would mean that $\mathcal{F}(\operatorname{sinc}(x))(\omega) = \pi v(-\omega) = \pi v(\omega)$. Thus $\mathcal{F}(\operatorname{sinc})(\omega) = \pi$ when $|\omega| < 1$ and $\mathcal{F}(\operatorname{sinc})(\omega) = 0$ when $|\omega| \ge 1$. This is a formal result since $\operatorname{sinc}(x)$ does *not* belong to $G(\mathbf{R})$, so our definition of the Fourier transform does not hold. However, the function $\operatorname{sinc}^2(x)$ does belong to $G(\mathbf{R})$, so there exists a Fourier transform of $u(x)$. Proceeding formally, we find that

$$\mathcal{F}(\operatorname{sinc}^2)(\omega) = \frac{1}{2\pi}\,\mathcal{F}(\operatorname{sinc}) * \mathcal{F}(\operatorname{sinc})(\omega) = \begin{cases} \dfrac{\pi}{2}\big(2 - |\omega|\big), & |\omega| < 2, \\ 0, & |\omega| \ge 2. \end{cases}$$

Why? Well, the procedure is analogous to the example we saw earlier. We need to calculate the convolution of two identical boxes, so symmetry should almost be enough to assume that it's a triangle but let's do the calculation. Let $F(\omega) = \mathcal{F}(\operatorname{sinc})(\omega)$.



So if $-2 < \omega < 0$, then

$$F * F(\omega) = \int_{-1}^{\omega+1} F(\xi)F(\omega - \xi)\, d\xi = \int_{-1}^{\omega+1} \pi^2\, d\xi = \pi^2(\omega + 2),$$

and if $0 < \omega < 2$, then

$$F * F(\omega) = \int_{\omega-1}^{1} F(\xi)F(\omega - \xi)\, d\xi = \int_{\omega-1}^{1} \pi^2\, d\xi = \pi^2(2 - \omega).$$

For $|w| > 2$ we have $F * F(\omega) = 0$.

So is this really the Fourier transform of $\operatorname{sinc}^2(x)$? One way of proving this is to actually use the inversion formula we derived above, which basically means that we take the Fourier transform of the function $V(\omega) = \pi(2 - |\omega|)/2$ for $|\omega| < 2$ (and zero elsewhere):

$$\mathcal{F}^{-1}\, V(\omega) = \frac{1}{2\pi}\int_{-2}^{2} \frac{\pi}{2}(2 - |\omega|)e^{i\omega x}\, d\omega = \cdots = \frac{1}{4}\,\frac{2 - e^{-i2x} - e^{i2x}}{x^2} = \frac{1}{2}\,\frac{1 - \cos 2x}{x^2} = \operatorname{sinc}^2(x).$$

This operation is allowed since it is clear that $V \in G(\mathbf{R})$ so the Fourier transform is defined as before. So we now know that $u \in G(\mathbf{R})$ is continuous and differentiable, and thus

$$\mathcal{F}^{-1}\,\mathcal{F}\, u(x) = u(x).$$

Moreover, we just showed that

$$(\mathcal{F}^{-1} V(\omega))(x) = u(x),$$

so

$$\mathcal{F}^{-1} \mathcal{F} u(x) = (\mathcal{F}^{-1} V(\omega))(x).$$

Does this mean that $\mathcal{F} u(\omega) = V(\omega)$? It actually does since $\mathcal{F} u$ and V are continuous, but we need the uniqueness result that we will prove next lecture. Why can't we use the one we derived in this lecture? Well, it's not clear that $D^{\pm} \mathcal{F} u$ exists, we only know that $\mathcal{F} u$ is uniformly continuous.

7.3.4 Properties of the Convolution Product

The convolution operation (on $L^1(\mathbf{R})$) behaves like we expect of a product in that it has the following properties.

Theorem. Suppose that $u, v, w \in G(\mathbf{R})$. Then the convolution has the following properties.

(i) Associative: $(u * v) * w(x) = u * (v * w)(x)$.

(ii) Distributive: $(u + v) * w(x) = u * w(x) + v * w(x)$.

(iii) Commutative: $u * v(x) = v * u(x)$.

Proof. Since the convolution of functions from $G(\mathbf{R})$ are mapped to the product of their respective Fourier transforms, all of these properties follow from the fact that they hold for the regular product. After showing the corresponding identities for the Fourier transform, we take the inverse transform to obtain the desired result, at least at all points where the factors are continuous. We need to use a fact that we will show on the next lecture here, where we show that if $u \in G(\mathbf{R})$ is continuous, then we obtain that $\mathcal{F}^{-1} \mathcal{F} u = u$. □

Note that these properties are only guaranteed when the elements belong to $G(\mathbf{R})$. It is also quite possible to directly prove that these properties hold from the definition of the convolution. An interesting question is if there is a unit for the convolution? That is, is there some element δ such that $u * \delta = u$ for all u? It turns out that this is *not* possible with $\delta \in L^1(\mathbf{R})$, but moving over to *distributions*, we can consider the Dirac impulse "function."

7.4 Plancherel's Formula

Recall that the space $L^2(\mathbf{R})$ consists of those functions $u \colon \mathbf{R} \to \mathbf{C}$ such that

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty.$$

It is true that if $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\mathcal{F} u \in L^2(\mathbf{R})$. This fact is far from trivial, but the following result holds.

Plancherel's theorem

Theorem. Suppose that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F}\, u(\omega)|^2 \, d\omega.$$

Analogously with the case for Fourier series (using the polarization identity), we can obtain the following generalization.

Plancherel's (generalized) formula

Theorem. Suppose that $u, v \in G(\mathbf{R}) \cap L^2(\mathbf{R})$. Then

$$\int_{-\infty}^{\infty} u(x)\, \overline{v(x)} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\, u(\omega)\, \overline{\mathcal{F}\, v(\omega)} \, d\omega.$$

The proof of Plancherel's identity follows from Parseval's using the same polarization identity that was used for the corresponding proof for Fourier series. So we focus on proving Parseval's identity.

Proof. The first question is that it is not clear *a priori* that the integrals involved are defined. Remember that the Fourier transform $\mathcal{F}(u)$ is uniformly bounded by the L^1-norm of u, but what about the L^2-norm of $\mathcal{F}(u)$?

To attack this problem, we first assume that $u, v \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ are twice continuously differentiable (meaning of class $C^2(\mathbf{R})$) and have *compact support* (meaning basically that the functions are zero outside of a compact set, say $[-M, M]$ in our case). Then it is clear that $\mathcal{F}(u'')(\omega) = -\omega^2 \mathcal{F}\, u(\omega)$, so

$$|\mathcal{F}\, u(\omega)| = \left| \frac{\mathcal{F}(u'')(\omega)}{\omega^2} \right| \leq \frac{C}{|\omega|^2}, \quad \omega \neq 0,$$

where $C > 0$ exists due to the fact that we have the uniform bound

$$|\mathcal{F}(u'')(\omega)| \leq \|u''\|_{L^1(\mathbf{R})} < \infty$$

and u'' is continuous and $u''(x) = 0$ for $|x| > M$ for some constant M. Since also $\mathcal{F}(u)$ is continuous, this implies that $\mathcal{F}(u) \in L^1(\mathbf{R})$ (and obviously also $L^2(\mathbf{R})$). Analogously, it follows that $\mathcal{F}\, v \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Moreover, $\mathcal{F}^{-1} \mathcal{F}\, v = v$ (recall that v is also continuous). Then

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x)\overline{v(x)} \, dx &= \int_{-\infty}^{\infty} u(x)\overline{\mathcal{F}^{-1}(\mathcal{F}(v))(x)} \, dx \\
&= \int_{-\infty}^{\infty} u(x)\overline{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(v)(\omega) e^{i\omega x} \, d\omega} \, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x) \int_{-\infty}^{\infty} \overline{\mathcal{F}(v)(\omega)} e^{-i\omega x} \, d\omega \, dx \\
&= /\text{ Fubini } / = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(x) e^{-i\omega x} \, dx \right) \overline{\mathcal{F}(v)(\omega)} \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(u)(\omega)\, \overline{\mathcal{F}(v)(\omega)} \, d\omega,
\end{aligned}$$

which implies Parseval's formula (for functions in $C^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^2(\mathbf{R})$).
So the next question becomes if we can somehow approximate — in some useful sense — a general function u by something in C^2. And the answer is yes, although we defer the proof until the end of the lecture. For any $\epsilon > 0$, there exists a function $v \in C^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$\|u - v\|_{L^1(\mathbf{R})} < \epsilon \qquad \text{and} \qquad \|u - v\|_{L^2(\mathbf{R})} < \epsilon.$$

This means that we can choose a sequence $v_1, v_2, v_3, \ldots$ such that $v_k \to u$ in both $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$ (norm convergence). To simplify (without loss of generality as it turns out), we will only prove Parseval's identity (Plancherel's formula follows as stated previously). So let u belong to $G(\mathbf{R}) \cap L^2(\mathbf{R})$. First we prove that $\mathcal{F} u \in L^2(\mathbf{R})$. To this end, let $\Omega_R = [-R, R]$ and observe that

$$\| \mathcal{F} u\|_{L^2(\Omega_R)} \leq \| \mathcal{F} u - \mathcal{F} v_k\|_{L^2(\Omega_R)} + \| \mathcal{F} v_k\|_{L^2(\Omega_R)} \leq \| \mathcal{F} u - \mathcal{F} v_k\|_{L^2(\Omega_R)} + K, \tag{7.2}$$

where if A is a reasonable set (like a union of intervals),

$$\|w\|_{L^2(A)} := \left(\int_A |w(x)|^2 \, dx \right)^{1/2},$$

and $K > 0$ is some constant such that

$$K^2 \geq 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx = \int_{-\infty}^{\infty} | \mathcal{F} v_k(\omega)|^2 \, d\omega \geq \| \mathcal{F} v_k\|^2_{L^2(\Omega_R)}, \quad \text{for all } k = 1, 2, 3, \ldots.$$

This is possible since $\|v_k\|_{L^2(\mathbf{R})} \to \|u\|_{L^2(\mathbf{R})}$ by continuity, so the sequence of norms must be bounded. Indeed, the continuity of the norm is true in general: for any normed linear space X, the function $\| \cdot \| \colon X \to [0, \infty[$ is continuous due to the (reverse) triangle inequality:

$$|\|u\| - \|v\|| \leq \|u - v\|,$$

so for any $\epsilon > 0$, if $\|u - v\| < \delta = \epsilon$, then $|\|u\| - \|v\|| < \epsilon$. Obviously, this implies that also $\| \cdot \|^\alpha$ is continuous on X for any $\alpha > 0$.
Note also that K in (7.2) is independent of R. Now, since

$$\sup_{\omega \in \mathbf{R}} | \mathcal{F}(u - v_k)(\omega)| \leq \|u - v_k\|_{L^1(\mathbf{R})},$$

we obtain that

$$\begin{aligned} \| \mathcal{F} u - \mathcal{F} v_k\|_{L^2(\Omega_R)} &= \left(\int_{-R}^{R} | \mathcal{F}(u - v_k)(\omega)|^2 \, d\omega \right)^{1/2} \leq \left(\int_{-R}^{R} \| \mathcal{F}(u - v_k)\|^2_\infty \, d\omega \right)^{1/2} \\ &\leq \|u - v_k\|_{L^1(\mathbf{R})} \sqrt{2R} \to 0, \end{aligned} \tag{7.3}$$

as $k \to \infty$ for any $R > 0$. Letting $k \to \infty$ also completes the proof that $\mathcal{F} u \in L^2(\mathbf{R})$ since the bound is independent of R so we can let $R \to \infty$ after letting $k \to \infty$ (the order here is important).
We can now consider the following expression, where the integrals are convergent by the argument above. So, by the triangle inequality,

$$\begin{aligned} \left| 2\pi \int_{-\infty}^{\infty} |u(x)|^2 \, dx - \int_{-\infty}^{\infty} | \mathcal{F} u(\omega)|^2 \, d\omega \right| \leq & \left| 2\pi \int_{-\infty}^{\infty} |u(x)|^2 \, dx - 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx \right| \\ & + \left| 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2 \, dx - \int_{-\infty}^{\infty} | \mathcal{F} u(\omega)|^2 \, d\omega \right|. \end{aligned} \tag{7.4}$$

Note that

$$2\pi \int_{-\infty}^{\infty} |u(x)|^2\,dx - 2\pi \int_{-\infty}^{\infty} |v_k(x)|^2\,dx = 2\pi\left(\|u\|_2^2 - \|v_k\|_2^2\right) \to 0, \text{ as } k \to \infty, \tag{7.5}$$

since $\|v_k\|_2 \to \|u\|_2$. Moreover, since $v_k \in C^2 \cap G(\mathbf{R}) \cap L^2(\mathbf{R})$, it is true that

$$2\pi \int_{-\infty}^{\infty} |v_k(x)|^2\,dx = \int_{-\infty}^{\infty} |\mathcal{F}\, v_k(\omega)|^2\,d\omega,$$

so

$$2\pi \int_{-\infty}^{\infty} |v_k(x)|^2\,dx - \int_{-\infty}^{\infty} |\mathcal{F}\, u(\omega)|^2\,d\omega = \|\mathcal{F}\, v_k\|_2^2 - \|\mathcal{F}\, u\|_2^2.$$

We want to show that $\|\mathcal{F}\, v_k\|_2 \to \|\mathcal{F}\, u\|_2$, and by the (reverse) triangle inequality we have

$$\left|\|\mathcal{F}\, v_k\|_2 - \|\mathcal{F}\, u\|_2\right| \le \|\mathcal{F}\, v_k - \mathcal{F}\, u\|_2 = \|\mathcal{F}(v_k - u)\|_2 = \left(\int_{-\infty}^{\infty} |\mathcal{F}(v_k-u)(\omega)|^2\,d\omega\right)^{1/2}.$$

Recalling that the Fourier transform maps $G(\mathbf{R})$-functions into uniformly bounded functions, it is true that

$$|\mathcal{F}(v_k - u)(\omega)| \le \int_{-\infty}^{\infty} |v_k - u|\,dx,$$

where the right-hand side tends to zero (uniformly in ω). To exploit this, we need to split the integral into two parts before letting $k \to \infty$. Note that $\|\mathcal{F}\, v_k\|_2 = \sqrt{2\pi}\|v_k\|_2 \to \sqrt{2\pi}\|u\|_2$ implies that there exists a number N such that

$$\|\mathcal{F}\, v_k - \mathcal{F}\, v_n\|_2 < \frac{\epsilon}{3}, \quad k, n \ge N.$$

Let $n \ge N$ be fixed and choose $R > 0$ such that

$$\int_{|\omega|>R} |\mathcal{F}\, v_n(\omega)|^2\,d\omega < \frac{\epsilon^2}{9} \qquad \text{and} \qquad \int_{|\omega|>R} |\mathcal{F}\, u(\omega)|^2\,d\omega < \frac{\epsilon^2}{9}. \tag{7.6}$$

This is possible since $\mathcal{F}\, v_k, \mathcal{F}\, u \in L^2(\mathbf{R})$. Now,

$$\|\mathcal{F}(v_k-u)\|_{L^2(\mathbf{R})} \le \|\mathcal{F}(v_k-u)\|_{L^2(\Omega_R)} + \|\mathcal{F}(v_k-u)\|_{L^2(\Omega_R^c)},$$

and for any $R > 0$, $\|\mathcal{F}(v_k-u)\|_{L^2(\Omega_R)} \to 0$ due to (7.3). Furthermore,

$$\begin{aligned}\|\mathcal{F}(v_k-u)\|_{L^2(\Omega_R^c)} &\le \|\mathcal{F}(v_k-v_n)\|_{L^2(\Omega_R^c)} + \|\mathcal{F}(v_n-u)\|_{L^2(\Omega_R^c)}\\ &\le \|\mathcal{F}(v_k-v_n)\|_{L^2(\mathbf{R})} + \|\mathcal{F}(v_n-u)\|_{L^2(\Omega_R^c)} < \frac{\epsilon}{3} + \|\mathcal{F}(v_n-u)\|_{L^2(\Omega_R^c)}\end{aligned}$$

and

$$\|\mathcal{F}(v_n-u)\|_{L^2(\Omega_R^c)} \le \|\mathcal{F}\, v_n\|_{L^2(\Omega_R^c)} + \|\mathcal{F}\, u\|_{L^2(\Omega_R^c)} < \frac{2\epsilon}{3}$$

because of (7.6). Hence

$$\|\mathcal{F}(v_k-u)\|_{L^2(\mathbf{R})} \le 2R\|v_k-u\|_{L^1(\mathbf{R})} + \epsilon.$$

Letting $k \to \infty$ we find that $\|\mathcal{F}(v_k-u)\|_2 < \epsilon$ and since $\epsilon > 0$ was arbitrary, this proves that $\|\mathcal{F}\, v_k\|_2 \to \|\mathcal{F}\, u\|_2$ as $k \to \infty$. This also completes the proof that the right-hand side of (7.4) can be made arbitrarily small. □

Example

Calculate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}\,d\omega$.

Solution. We observe that $U(\omega) = \dfrac{1}{1+\omega^2}$ is the Fourier transform of $u(x) = \dfrac{1}{2}\,e^{-|x|}$. Since it is clear that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, Plancherel's formula implies that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}\,d\omega &= 2\pi \int_{-\infty}^{\infty} \left(\frac{1}{2}\,e^{-|x|}\right)^2 dx = 4\pi \int_0^{\infty} \left(\frac{1}{2}\,e^{-|x|}\right)^2 dx = \pi \int_0^{\infty} e^{-2x}\,dx \\ &= \pi \left[-\frac{e^{-2x}}{2}\right]_0^{\infty} = \frac{\pi}{2}.\end{aligned}$$

7.5 Proof That $\int_0^\infty \operatorname{sinc}(x)\,dx = \frac{\pi}{2}$

First we prove that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx \tag{7.7}$$

is convergent. The idea is that if we know this, we can choose a particular way for the upper limit to approach infinity (and be sure that this is the correct value).
Note that since $\sin(x)/x$ is bounded and continuous (the limit when $x \to 0$ is 1), it is clear that

$$\int_0^{\pi} \frac{\sin x}{x}\,dx$$

is convergent. Now, using integration by parts we obtain

$$\int_{\pi}^{b} x^{-1}\sin x\,dx = \left[-x^{-1}\cos x\right]_{\pi}^{b} - \int_{\pi}^{b} \frac{\cos x}{x^2}\,dx.$$

The integral in the right-hand side is absolutely convergent since

$$\int_{\pi}^{b} \left|\frac{\cos x}{x^2}\right| dx \le \int_{\pi}^{b} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{\pi}^{b} = \frac{1}{\pi} - \frac{1}{b} \to \frac{1}{\pi}$$

as $b \to \infty$ (the exact number is not important and similarly the number π is arbitrary). So the conclusion is that (7.7) is convergent.
Since (7.7) is convergent, we can find it's value by the following calculation:

$$\begin{aligned}\int_0^{\infty} \frac{\sin x}{x}\,dx &= \lim_{\mathbf{Z}\ni m\to\infty} \int_0^{(m+1/2)\pi} \frac{\sin x}{x}\,dx \\ &= \Big/\ t = \frac{x}{m+1/2}\ \Big/ = \lim_{\mathbf{Z}\ni m\to\infty} \int_0^{\pi} \frac{\sin(t(m+1/2))}{t}\,dt \\ &= \lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2} \int_0^{\pi} \frac{2\sin(t/2)}{t} D_m(t)\,dt\end{aligned}$$

where $D_m(t)$ is the Dirichlet kernel on $[-\pi,\pi]$, that is

$$D_m(t) = \sum_{k=-m}^{m} e^{-ikt} = \frac{\sin(t(2m+1)/2)}{\sin(t/2)};$$

see Lecture 3. Moreover, the convergence result from Lecture 3 shows that if $u \in E'[-\pi,\pi]$, then

$$\lim_{m\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t+x)D_m(t)\,dt = \frac{u(x^+)+u(x^-)}{2}.$$

So letting $u(t) = \dfrac{2\sin(t/2)}{t}$ for $0 \le t \le \pi$ and $u = 0$ for $-\pi < t < 0$ (and extended periodically), we observe that obviously $u \in E$ and we see that

$$\begin{aligned} D^+u(0) &= \lim_{h\to 0^+} \frac{u(h)-u(0)}{h} = \lim_{h\to 0^+} \frac{2\sin(h/2)/h - 1}{h} = \lim_{h\to 0^+} \frac{1}{h^2}\left(2\sin(h/2) - h\right) \\ &= \lim_{h\to 0^+} \frac{1}{h^2}\Big(2\left(h/2 + O(h^3)\right) - h\Big) = \lim_{h\to 0^+} O(h) = 0. \end{aligned}$$

Obviously $D^-u(0) = 0$. Since $u(0^+) = 1$ and $u(0^-) = 0$, we therefore obtain that

$$\lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2}\int_0^{\pi} \frac{2\sin(t/2)}{t} D_m(t)\,dt = \pi \lim_{\mathbf{Z}\ni m\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)D_m(t)\,dt = \pi\,\frac{1+0}{2} = \frac{\pi}{2}.$$

7.5.1 ...but it is not absolutely convergent

Note though, that (7.7) is *not* absolutely convergent. We can see this by rewriting as a series of partial integrals:

$$\int_{\pi}^{\infty} \frac{|\sin x|}{|x|}\,dx = \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \ge \sum_{k=1}^{\infty} \frac{1}{k\pi}\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k} = \infty,$$

since $\displaystyle\int_0^{\pi} |\sin x|\,dx = 2$ (this is the same for every interval $[k\pi,(k+1)\pi]$).

7.6 An Approximation Result

So this is going to be fairly similar to what we did in lecture 5, but instead of waving our hands, let's go through the details.

Theorem. Suppose that $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$ and let $\epsilon > 0$. Then there exists a function v in $C^2(\mathbf{R})$ such that the following holds.

(i) There exists an interval $[-M, M]$ such that $v(x) = 0$ for $|x| > M$.

(ii) $\displaystyle\int_{-\infty}^{\infty} |u(x) - v(x)|^2 \, dx < \epsilon^2$.

(iii) $\displaystyle\int_{-\infty}^{\infty} |u(x) - v(x)| \, dx < \epsilon$.

Proof. To produce such a function v, we will use the fact that u and $|u|^2$ are absolutely integrable (in the Riemann sense) to find a partition where any Riemann sum is close enough to the integral. Before doing this, lets fix so we have *compact support*. We do this by observing that since u and $|u|^2$ are absolutely integrable on $\mathbf{R}$, there exists a number $L > 0$ such that

$$\max\left\{\int_{-\infty}^{-L} |u(x)| \, dx + \int_{L}^{\infty} |u(x)| \, dx,\ \int_{-\infty}^{-L} |u(x)|^2 \, dx + \int_{L}^{\infty} |u(x)|^2 \, dx\right\} < \min\left\{\frac{\epsilon}{3}, \frac{\epsilon^2}{18}\right\}.$$

Now, on $[-L, L]$, we choose a partition

$$x_0 = -L < x_1 < x_2 < \cdots < x_n = L$$

such that u is continuous on each $]x_k, x_{k+1}[$ and

$$\left|\int_{-L}^{L} u(x) \, dx - \sum_{k=0}^{n-1} c_k (x_{k+1} - x_k)\right| < \frac{\epsilon}{3},$$

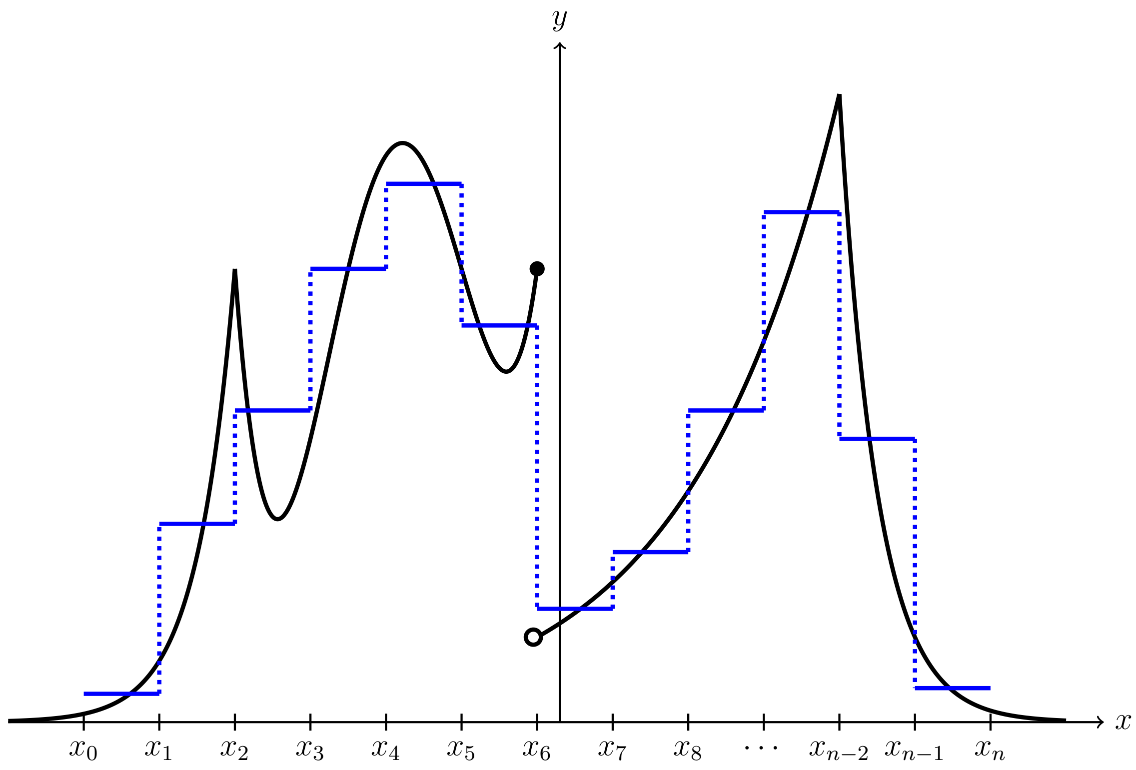
where $c_k = u(\xi_k)$ for some $\xi_k \in [x_k, x_{k+1}]$. Let $\zeta(x) = c_k$ when $x_k \leq x < x_{k+1}$, $k = 0, 1, 2, \ldots$ and zero elsewhere.

Note that by the uniform continuity of u on each $[x_i, x_{i+1}]$ (after possible redefinition at the end points), it is true that for any $\epsilon > 0$, there is a $\delta_i > 0$ such that

$$x, y \in [x_i, x_{i+1}] : |x - y| < \delta_i \quad \Rightarrow \quad |u(x) - u(y)| < \min\left\{\frac{\epsilon}{6L}, \frac{\epsilon}{\sqrt{18L}}\right\}.$$

We therefore choose $\delta = \min\{\delta_i\}$ and since clearly $\delta > 0$, it is possible to refine the partition $\{x_i\}_{i=0}^{n}$ of $[-L, L]$ such that $|x_{i+1} - x_i| < \delta$, $i = 0, 1, 2, \ldots, n-1$.

Graphically, we could have something like this.



From this it follows that

$$|u(x) - \zeta(x)| = |u(x) - c_k| \le \min\left\{\frac{\epsilon}{6L}, \frac{\epsilon}{\sqrt{36L}}\right\}, \quad x_i < x < x_{i+1},$$

since $c_k = u(\xi_k)$ for some ξ_k such that $x_k < \xi_k \le x_{k+1}$. The inequality might not hold at the end-points, but this does not matter for the integral. This implies that

$$\int_{x_i}^{x_{i+1}} |u(x) - \zeta(x)|\, dx \le \frac{\epsilon}{6L} |x_{i+1} - x_i|, \quad i = 0, 1, 2, \ldots, n-1$$

and

$$\int_{x_i}^{x_{i+1}} |u(x) - \zeta(x)|^2\, dx \le \frac{\epsilon^2}{36L} |x_{i+1} - x_i|, \quad i = 0, 1, 2, \ldots, n-1,$$

so

$$\begin{aligned}
\|u - \zeta\|^2_{L^2(\mathbf{R})} &= \int_{-\infty}^{\infty} |u(x) - \zeta(x)|^2\, dx \\
&= \int_{-\infty}^{-L} |u(x)|^2\, dx + \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |u(x) - \zeta(x)|^2\, dx + \int_{L}^{\infty} |u(x)|^2\, dx \\
&\le \frac{\epsilon^2}{18} + \sum_{k=0}^{n-1} \frac{\epsilon^2}{36L} |x_{i+1} - x_i| = \frac{\epsilon^2}{9}
\end{aligned}$$

and

$$\begin{aligned}
\|u - \zeta\|_{L^1(\mathbf{R})} &= \int_{-\infty}^{\infty} |u(x) - \zeta(x)|\, dx \\
&= \int_{-\infty}^{-L} |u(x)|\, dx + \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |u(x) - \zeta(x)|\, dx + \int_{L}^{\infty} |u(x)|\, dx \\
&\le \frac{\epsilon}{6} + \sum_{k=0}^{n-1} \frac{\epsilon}{6L} |x_{i+1} - x_i| = \frac{\epsilon}{3}
\end{aligned}$$

So how do we turn this into something that's twice differentiable? We will proceed similar to what we did in lecture 5, but a straight line will not do. Suppose we have two constant segments, one defined as 0 on $[-1, 0]$ and one defined as 1 on $[1, 2]$. Can we join these segments smoothly? Sure we can, in a lot of different ways. For our purpose, we need something of class C^2, so twice continuously differentiable. The most straight forward idea is probably to match a polynomial at the end points while making certain that also the derivatives match. Let $\eta(x)$ be such a polynomial. We want the following to hold:

$$\eta(0) = 0, \quad \eta(1) = 1, \quad \eta'(0) = 0, \quad \eta'(1) = 0, \quad \eta''(0) = 0, \quad \eta''(1) = 0.$$

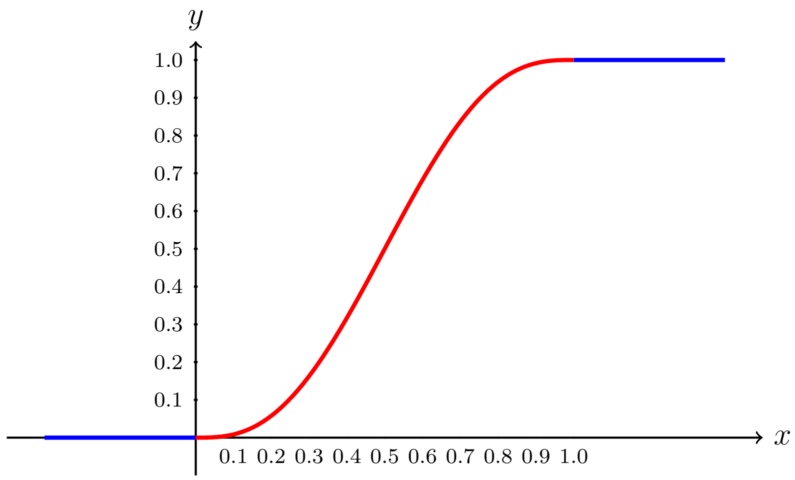
So six restrictions. Using a fifth degree polynomial as ansatz, we find that

$$\eta(x) = x^5 - 15x^4 + 10x^3.$$

Some basic analysis shows that there are no extreme values on $]0, 1[$ so the maximum and minimum are attained at the end points, which is nice since that means that $0 \leq \eta(x) \leq 1$ on $[0, 1]$. Let's make the following definition:

$$\eta(x) = \begin{cases} 0, & x < 0, \\ 6\,x^5 - 15\,x^4 + 10\,x^3, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

What we now have accomplished can be seen in the figure below.



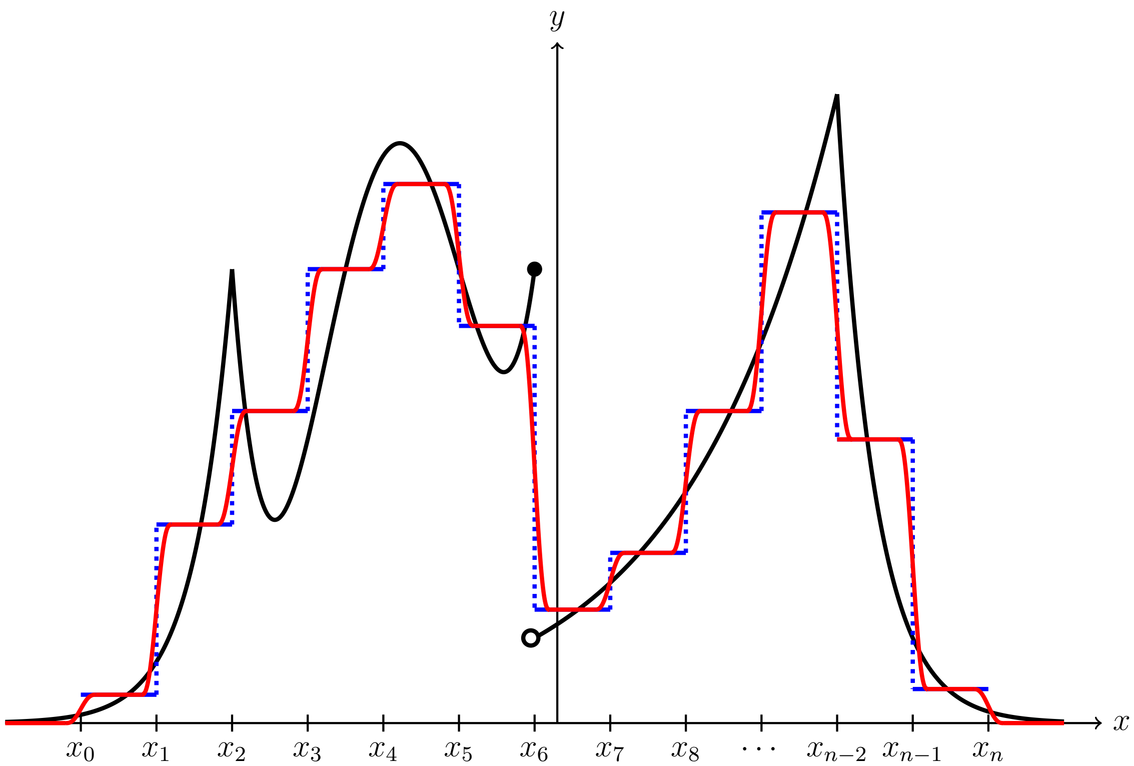
We can use this function in the following way, scaling and translating as needed. Choose a $\delta > 0$ such that (yeah yeah..)

$$\delta < \min\left\{\frac{\epsilon}{12K(n+1)}, \frac{\epsilon^2}{72K^2(n+1)}\right\},$$

where K is some number such that $|\zeta(x)| \leq K$ for all $x \in [-L, L]$. Define v such that $v(x) = c_k$ when $x_k + \delta \leq x \leq x_{k+1} - \delta$ and for $x_k - \delta < x < x_k + \delta$, we use the function

$$c_k + (c_{k+1} - c_k)\eta\left(\frac{x - (x_k - \delta)}{2\delta}\right).$$

The result can be seen in the graph below.



Note that $v(x) = \zeta(x)$ for most of $\mathbf{R}$, so

$$\|v - \zeta\|^2_{L^2(\mathbf{R})} = \int_{-\infty}^{\infty} |v(x) - \zeta(x)|^2 \, dx = \sum_{k=0}^{n} \int_{x_k-\delta}^{x_k+\delta} |v(x) - \zeta(x)|^2 \, dx$$
$$\leq 8(n+1)K^2\delta \leq \frac{\epsilon^2}{9},$$

where we used the rough estimate $|v(x) - \zeta(x)| \leq 2K$ on $[-L, L]$, which holds if $|u(x)| \leq K$ (which implies that $|\zeta(x)| \leq K$ as well).
Similarly, we obtain that

$$\|v - \zeta\|_{L^1(\mathbf{R})} = \int_{-\infty}^{\infty} |v(x) - \zeta(x)| \, dx = \sum_{k=0}^{n} \int_{x_k-\delta}^{x_k+\delta} |v(x) - \zeta(x)| \, dx$$
$$\leq 4(n+1)K\delta \leq \frac{\epsilon}{3}.$$

Chapter 8

Uniqueness

"Consider that a divorce!"
—Douglas Quaid

8.1 Uniqueness

Similar to where we were in Lecture 5 for Fourier series, we know find ourselves in a similar spot with regards to the Fourier transform. Indeed, we have seen conditions for when the Fourier transform exists and we have seen conditions for when we can find the inverse (analogously to when the Fourier series converges "correctly").

Question. Suppose that $u, v \in G(\mathbf{R})$ has the Fourier transforms $\mathcal{F}\,u$ and $\mathcal{F}\,v$, respectively. If $\mathcal{F}\,u = \mathcal{F}\,v$, what can we say about the functions u and v? Are they equal? In what sense?

We will show that if $u, v \in G(\mathbf{R})$ and $\mathcal{F}\,u = \mathcal{F}\,v$, then $u(x) = v(x)$ wherever both u and v are continuous.

8.2 Cesàro Summation for Integrals

For our purposes, recall that we consider the *principal value* for the Fourier transform and its inverse, that is, integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx, \tag{8.1}$$

and that this might change for which functions f the integral is convergent. Now, we can obtain even better convergence by considering the mean value integral of the partial integrals, that is,

$$\lim_{M\to\infty} \frac{1}{M} \int_0^M \int_{-r}^{r} f(x)\,dx\,dr. \tag{8.2}$$

This is analogous to the mean value of the partial sums for the Cesàro summation for series. Similarly to that case, if the limit in (8.1) exists, then the limit in (8.2) exists as well and converges to the same value.

Indeed, let $I_r = \displaystyle\int_{-r}^{r} u(x)\,dx \to I$ be convergent and let $\epsilon > 0$. Then there exists $N > 0$ such that $|I_r - I| \leq \epsilon$ if $r \geq N$ and

$$\left|\frac{1}{M}\int_0^M I_r\,dr - I\right| = \left|\frac{1}{M}\int_0^M (I_r - I)\,dr\right| \leq \frac{1}{M}\int_0^N |I_r - I|\,dr + \frac{1}{M}\int_N^M |I_r - I|\,dr.$$

Observing that

$$\int_0^N |I_r - I|\,dr \le \int_0^N \int_{-r}^r |u(x)|\,dx\,dr + NI \le N\int_{-N}^N |u(x)|\,dx + NI < \infty,$$

we find that

$$\lim_{M\to\infty} \frac{1}{M}\int_0^N |I_r - I|\,dr = 0.$$

Since also

$$\frac{1}{M}\int_N^M |I_r - I|\,dr \le \frac{1}{M}\int_N^M \epsilon\,dr \le \frac{M-N}{M}\epsilon < \epsilon,$$

it must be true that

$$\lim_{M\to\infty} \frac{1}{M}\int_0^M I_r\,dr = I.$$

8.3 The Fejér Kernel for the Fourier Transform

We wish to investigate the limit

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^R \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega$$

and see if it exists, and if so, what the limit is (hoping for something similar to $u(x)$). To this end, let's consider the Cesàro means:

$$\begin{aligned}\frac{1}{M}\int_0^M \left(\frac{1}{2\pi}\int_{-r}^r \mathcal{F}\,u(\omega)e^{i\omega x}\,d\omega\right)dr &= \frac{1}{2\pi M}\int_{-M}^M \int_{|\omega|}^M \mathcal{F}\,u(\omega)e^{i\omega x}\,dr\,d\omega \\ &= \frac{1}{2\pi M}\int_{-M}^M \mathcal{F}\,u(\omega)\left(M - |\omega|\right)e^{i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-M}^M \mathcal{F}\,u(\omega)\left(1 - \frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega,\end{aligned}$$

where we changed the order of integration in the first equality. Now, writing out the definition of $\mathcal{F}\,u(\omega)$, we find that

$$\begin{aligned}\frac{1}{2\pi}\int_{-M}^M \mathcal{F}\,u(\omega)\left(1 - \frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega &= \frac{1}{2\pi}\int_{-M}^M \left(\int_{-\infty}^\infty u(t)e^{-it\omega}\,dt\right)\left(1 - \frac{|\omega|}{M}\right)e^{i\omega x}\,d\omega \\ &= \int_{-\infty}^\infty u(t)\frac{1}{2\pi}\int_{-M}^M \left(1 - \frac{|\omega|}{M}\right)e^{i\omega(x-t)}\,d\omega\,dt \\ &= \int_{-\infty}^\infty u(t)F_M(x-t)\,dt = \int_{-\infty}^\infty u(t+x)F_M(t)\,dt,\end{aligned}$$

where we used Fubini's theorem and where

$$F_M(t) = \frac{1}{2\pi}\int_{-M}^M \left(1 - \frac{|\omega|}{M}\right)e^{i\omega t}\,d\omega$$

is the Fejér kernel on the real line.

Theorem. For $x \neq 0$, we have

$$F_M(x) = \frac{1-\cos Mx}{\pi M x^2} = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2. \tag{8.3}$$

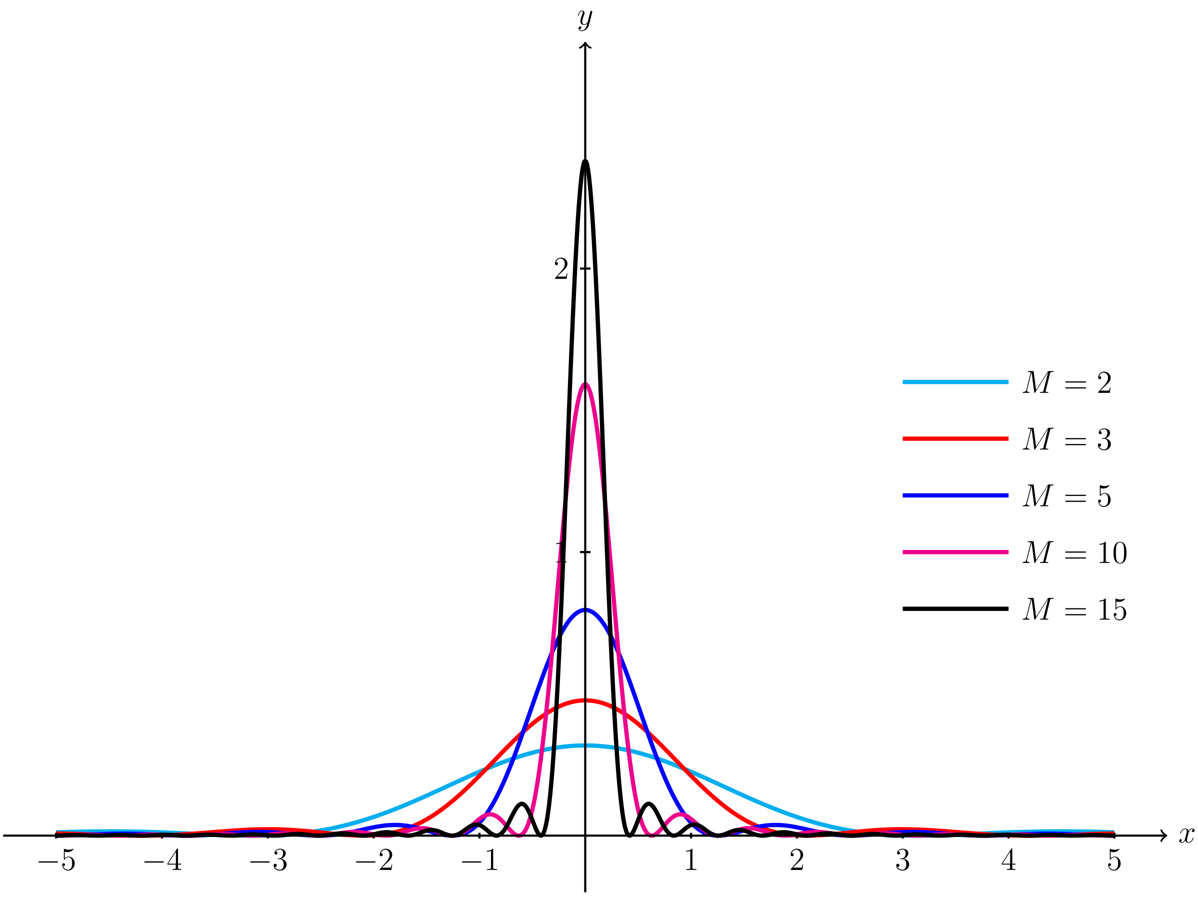
Proof. Using integration by parts, we obtain that for $x \neq 0$,

$$\begin{aligned}\int_{-M}^{M}\left(1-\frac{|w|}{M}\right)e^{i\omega x}\,d\omega &= \frac{1}{ix}\left(\left[\left(1-\frac{|w|}{M}\right)e^{i\omega x}\right]_{\omega=-M}^{\omega=M} + \frac{1}{M}\int_{-M}^{M}\operatorname{sgn}(\omega)e^{i\omega x}\,d\omega\right)\\ &= \frac{1}{M(ix)^2}\left(-\left[e^{i\omega x}\right]_{-M}^{0} + \left[e^{i\omega x}\right]_{0}^{M}\right) = \frac{1}{Mx^2}\left(1-e^{-iM\omega x}-e^{iMx}+1\right)\\ &= \frac{2-2\cos Mx}{Mx^2},\end{aligned}$$

so

$$F_M(x) = \frac{1}{M\pi}\,\frac{1-\cos Mx}{x^2}, \quad x \neq 0. \tag{8.4}$$

Since $2\sin^2 t = 1-\cos 2t$, $t \in \mathbf{R}$, the second formula above follows from (8.4). □



Properties of the Fejér kernel on the real line

Theorem.

(i) $F_M(x) \geq 0$ and F_M is an even function.

(ii) $\displaystyle\int_{-\infty}^{\infty} F_M(x)\,dx = 1.$

(iii) If $\tau > 0$, then $\displaystyle\lim_{M\to\infty} F_M(x) = 0$ uniformly for $|x| \geq \tau$.

(iv) $\displaystyle\int_{|x|\geq\tau} F_M(x)\,dx \to 0$ for any $\tau > 0$.

Proof.

(i) These properties are obvious from the definition.

(ii) To prove this identity, observe that if $\phi(x) = 1 - |x|$ for $|x| < 1$ and $\phi(x) = 0$ for $|x| \geq 1$, then (according to (8.3) above with $M = 1$ and ω replaced by $-\omega$)

$$\mathcal{F}\,\phi(\omega) = \int_{-1}^{1} (1-|x|)e^{-i\omega x}\,dx = \frac{2-2\cos\omega}{\omega^2}.$$

Now, since ϕ is continuous (at zero) and $D^{\pm}\phi(0)$ exists, we know that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2-2\cos\omega}{\omega^2}\,e^{i\omega\cdot 0}\,d\omega = \phi(0) = 1.$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} F_M(x)\,dx &= \int_{-\infty}^{\infty} \frac{1}{M\pi}\,\frac{1-\cos Mx}{x^2}\,dx \\ &= /\ t = Mx\ / = \int_{-\infty}^{\infty} \frac{1}{M\pi}\,\frac{1-\cos t}{(t/M)^2}\,\frac{dt}{M} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos t}{t^2}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2-2\cos t}{t^2}\,dt = 1.\end{aligned}$$

(iii) Observing that

$$|F_M(x)| = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2 \leq \frac{M}{2\pi}\left(\frac{2}{Mx}\right)^2 = \frac{2}{M\pi}\,\frac{1}{x^2}, \tag{8.5}$$

we see that

$$\sup_{|x|\geq\tau} |F_M(x)| \leq \frac{2}{M\pi}\sup_{|x|\geq\tau}\frac{1}{x^2} = \frac{2}{M\pi\tau^2} \to 0,$$

as $M \to \infty$. Hence we have uniform convergence for $|x| \geq \tau$ for any $\tau > 0$.

(iv) Furthermore, inequality (8.5) also implies that

$$\int_{\tau}^{\infty} F_M(x)\,dx \leq \frac{2}{M\pi}\int_{\tau}^{\infty}\frac{1}{x^2}\,dx = \frac{2}{M\pi\tau} \to 0,$$

as $M \to \infty$. The integral from $-\infty$ to $-\tau$ is handled analogously. □

Theorem. Suppose that $u \in G(\mathbf{R})$ (so u has right- and lefthand limits at $x \in \mathbf{R}$). Then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\, u(\omega) \left(1 - \frac{|\omega|}{R}\right) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

Proof. Since

$$\frac{1}{2\pi} \int_{-M}^{M} \mathcal{F}\, u(\omega) \left(1 - \frac{|\omega|}{M}\right) e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty} u(t+x) F_M(t)\, dt,$$

we start by proving that

$$\int_{0}^{\infty} (u(x+t) - u(x^+)) F_M(t)\, dt + \int_{-\infty}^{0} (u(x+t) - u(x^-)) F_M(t)\, dt \to 0,$$

as $M \to \infty$. This implies that the Fejér mean converges:

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} u(t) F_M(x-t)\, dt = \frac{u(x^+) + u(x^-)}{2}$$

since

$$\int_{-\infty}^{0} F_M(t)\, dt = \int_{0}^{\infty} F_M(t)\, dt = \frac{1}{2}.$$

Let $\epsilon > 0$. Since u has a right-hand limit at x, there is a $\delta > 0$ such that

$$0 < t < \delta \quad \Rightarrow \quad |u(x+t) - u(x^+)| < \epsilon.$$

We exploit this and the uniform convergence of F_M to obtain that

$$\begin{aligned}
\left| \int_{0}^{\infty} (u(x+t) - u(x^+)) F_M(t)\, dt \right| &\le \int_{0}^{\delta} \epsilon F_M(t)\, dt + \int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt \\
&\le \epsilon \int_{0}^{\infty} F_M(t)\, dt + \int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt \\
&\to \frac{\epsilon}{2}
\end{aligned}$$

as $M \to \infty$ since F_M converges uniformly to zero on $[\delta, \infty[$, so

$$\begin{aligned}
\int_{\delta}^{\infty} |u(x+t) - u(x^+)| F_M(t)\, dt &\le \left(\sup_{t\ge\delta} F_M(t) \right) \int_{\delta}^{\infty} |u(x+t)|\, dx + |u(x^+)| \int_{\delta}^{\infty} F_M(t)\, dt \\
&\le \left(\sup_{t\ge\delta} F_M(t) \right) \int_{-\infty}^{\infty} |u(x)|\, dx + |u(x^+)| \int_{\delta}^{\infty} F_M(t)\, dt \to 0,
\end{aligned}$$

as $M \to \infty$. The second integral is handled analogously. □

An immediate consequence of this theorem is the following uniqueness result.

Uniqueness

Corollary. Suppose that $u \in G(\mathbf{R})$ and $v \in G(\mathbf{R})$. If $\mathcal{F}\,u(\omega) = \mathcal{F}\,v(\omega)$ for every $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for every $x \in \mathbf{R}$ where both u and v are continuous.

Furthermore, the following corollary is clear since if an integral converges in the usual sense, then the Cesàro-means converge to the same value. This shows that the assumption that the onesided derivatives exist, which we used in the previous lecture, is not necessary. The inversion works anyway for functions in $G(\mathbf{R})$, provided that the limit exists.

Corollary. Suppose that $u \in G(\mathbf{R})$. Then

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \mathcal{F}\,u(\omega) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2},$$

whenever the limit exists.

This means that if the limit exists and u has right- and lefthand limits, then the inversion gives the expected result.

Chapter 9

The Unilateral Laplace Transform

"Here's Sub-Zero. Now... Plain Zero!"
—Ben Richards

9.1 The One Sided Laplace Transform

For reasonable functions, we make the following definition.

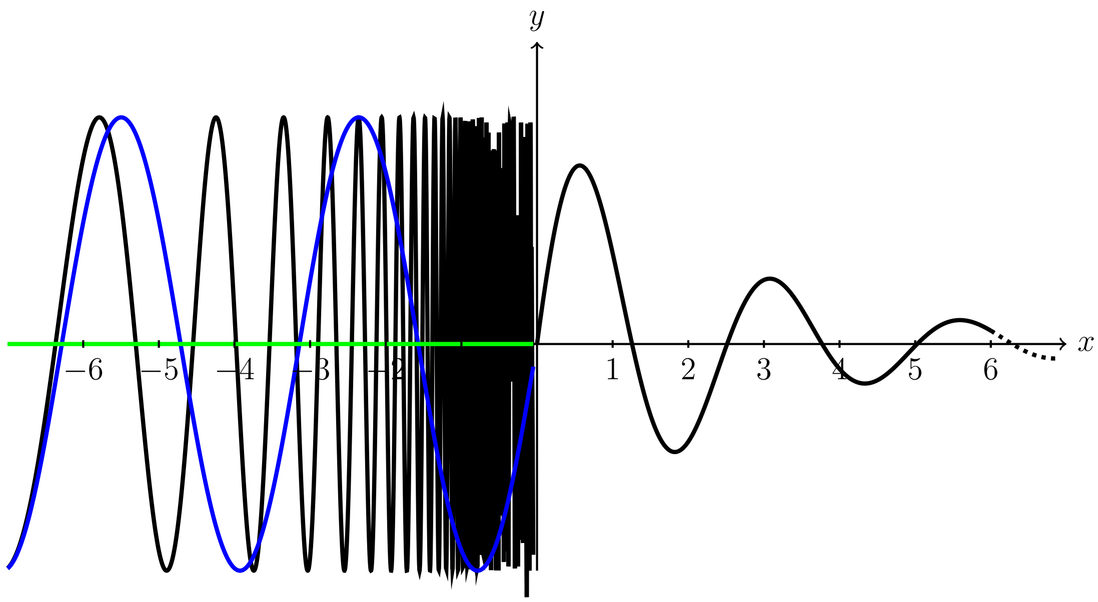
The Laplace transform

Definition. The Laplace transform of $u\colon [0,\infty[\to \mathbf{C}$ is given by

$$\mathcal{L}\,u(s) = \int_0^\infty u(t)e^{-st}\,dt,$$

for those $s \in \mathbf{C}$ where this integral is convergent.

Note that in this definition, we start integrating at $t = 0$. This means that whatever u does for $t < 0$, it is not in any way connected with $\mathcal{L}\,u(s)$. We say that $\mathcal{L}\,u(s)$ is the **one-sided** or **unilateral Laplace transform**.



Black, blue, green... doesn't matter, the Laplace transform will be the same. Therefore we often assume that $u(t) = 0$ for $t < 0$.
Why this restriction? Well, it does make the transform easier to handle. Secondly, there are a lot of applications where we consider the variable t to be *time*, so negative values are not very interesting. Indeed, we assume that something starts at $t = 0$. In other words, we consider *causal* systems. There is a two-sided version of the Laplace transform as well, which is useful in many instances, but in this course we will only use the version above.

Example

Suppose that $u(t) = e^{at}$, where $a \in \mathbf{C}$ is a constant. Show that $\mathcal{L}\,u(s) = \dfrac{1}{s-a}$, $\operatorname{Re} s > \operatorname{Re} a$.

Solution. We find that

$$\mathcal{L}\,u(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}, \quad \text{if } \operatorname{Re}(s-a) > 0.$$

As we can see in the example above, the Laplace transform exists if $\operatorname{Re} s > \operatorname{Re} a$. For a function that doesn't grow faster than e^{at}, the Laplace transform will exist at least for $\operatorname{Re} s > \operatorname{Re} a$ (provided that the integral exists). For our purposes, piecewise continuous functions will suffice. Let's make the following definition.

Exponential order (exponential growth)

Definition. We say that the piecewise continuous function $u\colon [0,\infty[$ is of exponential order (of order a) if there exists constants $a > 0$ and $K > 0$ such that $|u(t)| \le Ke^{at}$ for $t \ge 0$. The set of all such functions will be denoted by X_a.

Existence

Theorem. If $u \in X_a$ for some $a > 0$, then the Laplace transform $\mathcal{L}\,u(s)$ exists (at least) for $\operatorname{Re} s > a$. Furthermore,

$$\lim_{L\to\infty}\int_0^L u(t)\,e^{-st}\,dt = \mathcal{L}\,u(s)$$

uniformly and $\mathcal{L}\,u(s)$ is continuous.

Proof. Obviously

$$\left|u(t)e^{-st}\right| \le Ke^{at}|e^{-st}| = Ke^{at}e^{-t\operatorname{Re} s} = Ke^{-t(\operatorname{Re} s - a)},$$

so if $\operatorname{Re} s > a$ then

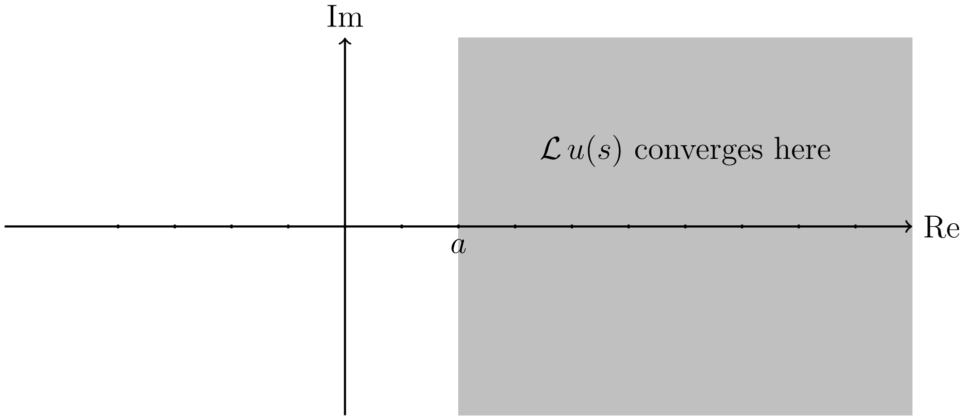
$$\int_0^\infty u(t)e^{-st}\,dt$$

converges absolutely. The fact that the convergence is uniform follows from the inequality above. Indeed, suppose that $\operatorname{Re} s \ge b > a$. Then

$$\sup_{\operatorname{Re} s\ge b}\left|\int_0^\infty u(t)\,e^{-st}\,dt - \int_0^L u(t)\,e^{-st}\,dt\right| \le \sup_{\operatorname{Re} s\ge b}\int_L^\infty \left|u(t)e^{-st}\right|\,dt$$
$$\le K\sup_{\operatorname{Re} s\ge b}\int_L^\infty e^{-t(\operatorname{Re} s-a)}\,dt = \frac{Ke^{-L(b-a)}}{b-a} \to 0,$$

as $L \to \infty$. Since the limit is uniform, we also obtain that $\mathcal{L}\, u(s)$ is continuous. □

So the region of convergence for the one-sided Laplace transform of functions from X_a typically looks like this.



9.2 Connection to the Fourier Transform?

The Fourier transform is basically a slice of the Laplace transform — where we let $s = i\omega$ — when we restrict the argument function to non-negative values. In other words, we only let s move along the imaginary axis and consider the function $f(t)H(t)$ where $H(t)$ is the Heaviside function. So if u is piecewise continuous and $\mathcal{L}\, u(s)$ exists and $s = \sigma + i\omega$, then

$$\mathcal{L}\, u(s) = \mathcal{F}\big(e^{-\sigma t} u(t)H(t)\big)(\omega).$$

This means that several things we did for the Fourier transform also holds for the Laplace transform, at least when it comes to the calculation of the transforms. The convergence results are different since we now allow exponential growth, but we can "move" the part corresponding to $\operatorname{Re} s$ to the argument function like above and use the corresponding result for the Fourier transform. We'll get back to this.
Qualitatively, one can say that the Fourier transform investigates frequency content in a function by decomposing the function into sinusoids while the Laplace transforms also investigates the amount of exponential growth/decay a function has.

9.3 Complex Differentiability and Analyticity

Since we're heading into a domain where $s \in \mathbf{C}$, we need to make sure everything is in order. So to this end, let's collect some facts we need. A complex valued function $u\colon \mathbf{C} \to \mathbf{C}$ is called differentiable if

$$u'(z) = \lim_{h \to 0} \frac{u(z+h) - u(z)}{h}$$

exists. The definition is basically the same as for the real case, but going back to the definition of a limit of a complex expression, it is clear that this two dimensional limit is more restrictive than that of the single variable case. Indeed, suppose that $u(z) = \alpha(x, y) + i\beta(x, y)$, where $z = x + iy$. Then the claim that u'_x and u'_y exist is *weaker* than claiming that $u'(z)$ defined as above exists. In fact, if u is differentiable then the components α and β satisfy the Cauchy-Riemann equations:

$$\alpha'_x = \beta'_y \quad \text{and} \quad \alpha'_y = -\beta'_x.$$

Equivalently, these equations can be phrased as

$$i\frac{\partial}{\partial x}u(x+iy) = \frac{\partial}{\partial y}u(x+iy).$$

There are results in the other direction as well. If u is continuous and has partial derivatives that satisfy Cauchy-Riemann's equations, then u is holomorphic (see below).
This topic is not something we need to dig that much deeper into. With the definition above, one can show that this complex derivative satisfies the same "rules" as in the real one-variable case, meaning that the product rule, chain rule, and so on works like expected.
We call a function **holomorphic** at a point z_0 if there is a neighborhood $B(z_0\,;\,\delta)$ of z_0 (meaning some open disc with z_0 as the center) such that f is differentiable for all points in this neighborhood. If one can choose all of $\mathbf{C}$ as the neighborhood, the function is usually referred to as **entire**. Something rather interesting happens here. Remember that the complex derivative requires more to exist than the partial derivatives, so what happens is that if u is holomorphic at z_0 then u is *infinitely differentiable* at z_0. Moreover, it turns out that the function is **analytic** at z_0, meaning that it's complex Taylor series converges to $u(z)$ for z in some neighborhood of z_0. So we have

$$u(z) \text{ holomorphic at } z_0 \quad \Leftrightarrow \quad u(z) = \sum_{k=0}^{\infty} \frac{u^{(k)}(z_0)}{k!}\,(z-z_0)^k$$

in some neighborhood of z_0. Holomorphic functions have several other very nice properties such as

(i) Cauchy's integral theorem: $\oint_\gamma u(z)\,dz = 0$ for any closed nice enough curve γ;

(ii) Cauchy's integral formula: $u(z) = \dfrac{1}{2\pi i}\oint_\gamma \dfrac{u(\zeta)}{\zeta - z}\,d\zeta$ for $z \in D$ and $\gamma = \partial D$, when u is holomorphic on the closed disc D. This means that the values of u inside D are completely described by the values on the boundary.

(iii) Holomorphic functions are *conformal* if locally invertible (preserving angles).

We could go on and discuss meromorphic functions (holomorphic except for some exception points where we have *poles*) and residue calculus, but that's way off course. We will not use these types of properties, even if some arguments could be made a lot more elegant that way. The power series stuff and complex derivatives are enough.

9.3.1 The Laplace Transform is Analytic

If $u \in X_a$, the Laplace transform $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s > a$. We can prove this in several ways, and perhaps the most clear one is the following.
Assume first that u is continuous. Let $s = \sigma + i\omega$ and suppose that $a < \sigma_0 < \operatorname{Re} s = \sigma$. Then

$$\mathcal{L}\,u(s) = \int_0^\infty e^{-(\sigma+i\omega)t}u(t)\,dt$$

exists. Put $G(\sigma,\omega,t) = e^{-(\sigma+i\omega)t}u(t)$, $\sigma > \sigma_0$, $t \geq 0$ and $\omega \in \mathbf{R}$. Noting that

$$G'_\sigma(\sigma,\omega,t) = -tG(\sigma,\omega,t) \quad \text{and} \quad G'_\omega(\sigma,\omega,t) = -itG(\sigma,\omega,t),$$

we see that both

$$\int_0^\infty G'_\sigma(\sigma,\omega,t)\,dt = -\int_0^\infty te^{-st}u(t)\,dt \quad \text{and} \quad \int_0^\infty G'_\omega(\sigma,\omega,t)\,dt = -i\int_0^\infty te^{-st}u(t)\,dt$$

converge uniformly since $|te^{-st}| \le Ce^{-\sigma_0 t}$ and the Laplace transform is uniformly convergent for $\operatorname{Re} s > a$. Moreover, if u is continuous then G, G'_σ and G'_ω are all continuous. By Leibniz' rule, this implies that the partial derivatives

$$\frac{\partial}{\partial\sigma}\mathcal{L}\,u(s) = -\,\mathcal{L}(tu(t))(s) \quad \text{and} \quad \frac{\partial}{\partial\omega}\mathcal{L}\,u(s) = -i\,\mathcal{L}(tu(t))(s)$$

exist and are continuous, which is nice in of itself. However this also shows that the *Cauchy-Riemann equations* hold for $\mathcal{L}\,u(s)$:

$$i\frac{\partial}{\partial\sigma}\mathcal{L}\,u(s) = \frac{\partial}{\partial\omega}\mathcal{L}\,u(s),$$

which proves that $\mathcal{L}\,u(s)$ is analytic at the point s, so $\mathcal{L}\,u(s)$ is analytic for $\operatorname{Re} s > a$ (since the partial derivatives as well as $\mathcal{L}\,u(s)$ were continuous). If u is only piecewise continuous, one can proceed as in the proof of $i\,\mathcal{F}(xu(x)) = U'(\omega)$ from lecture 6.
So while we won't use the analyticity of the Laplace transform directly in this course, the following result will prove useful.

Time multiplication

Theorem. Let $u \in X_a$. Then $\mathcal{L}(tu(t))(s) = -\dfrac{d}{ds}\mathcal{L}(u(t))(s)$, $\operatorname{Re} s > a$.

Proof. Similarly to the case with the Fourier transform, observe (formally) that

$$\begin{aligned}\frac{d}{ds}\mathcal{L}\,u(s) &= \frac{d}{ds}\int_0^\infty u(t)e^{-st}\,dt = \big/ \text{ Lebiniz's rule } \big/ = \int_0^\infty u(t)\frac{d}{ds}e^{-st}\,dt \\ &= \int_0^\infty -tu(t)e^{-st}\,dt = -\,\mathcal{L}(tu(t))(s),\end{aligned}$$

where Leibniz's rule is applicable due to the argument above.

9.4 Rules for the Laplace Transform

The fact that the Laplace transform is an integral immediately proves that it is a linear operator.

Linearity

Theorem. If $a, b \in \mathbf{C}$ are constants, then $\mathcal{L}(au(t) + bv(t)) = a\,\mathcal{L}\,u + b\,\mathcal{L}\,v$, whenever $\mathcal{L}\,u$ and $\mathcal{L}\,v$ exists.

Example

Find the Laplace transforms of $\sin t$, $\cos t$ and $t\cos t$.

Solution. By Euler's equations, we obtain that

$$\mathcal{L}(\sin t)(s) = \mathcal{L}\left(\frac{e^{it}-e^{-it}}{2i}\right) = \frac{1}{2i}\left(\mathcal{L}(e^{it}) - \mathcal{L}(e^{-it})\right) = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{2i}\frac{2i}{s^2+1}$$
$$= \frac{1}{s^2+1},$$

for $\operatorname{Re} s > 0$. Similarly, it follows that

$$\mathcal{L}(\cos t)(s) = \frac{s}{s^2+1},$$

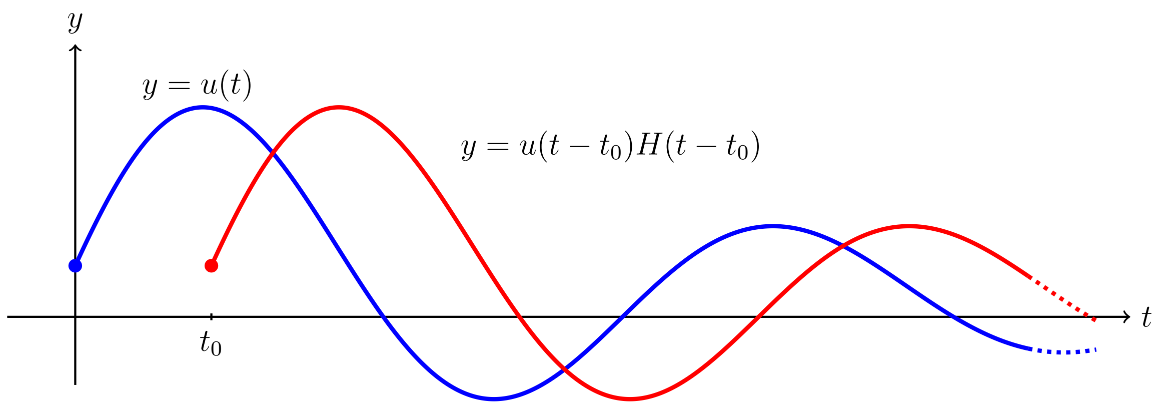
for $\operatorname{Re} s > 0$. To find the Laplace transform of $t\cos t$, we note that

$$\mathcal{L}(t\cos t)(s) = -\frac{d}{ds}\mathcal{L}(\cos t)(s) = -\frac{d}{ds}\frac{s}{s^2+1} = -\frac{s^2+1-2s^2}{(s^2+1)} = \frac{s^2-1}{(s^2+1)^2}.$$

Translation (time shift)

Theorem. If $t_0 > 0$ and $U(s) = \mathcal{L}\,u(s)$, then $\mathcal{L}(u(t-t_0)H(t-t_0))(s) = e^{-st_0}U(s)$.

The term $H(t-t_0)$ is important since we are working with the unilateral Laplace transform.



Notice the difference with the expression $u(t)H(t-t_0)$ (how would this look?).
Proof. A simple substitution shows that

$$\mathcal{L}(u(t-t_0)H(t-t_0))(s) = \int_{t_0}^{\infty} u(t-t_0)e^{-st}\,dt = /\ y = t - t_0\ / = \int_0^{\infty} u(y)e^{-s(y+t_0)}\,dy$$
$$= e^{-st_0}\int_0^{\infty} u(y)e^{-sy}\,dy = e^{-st_0}\,\mathcal{L}(u(t))(s).$$

Scaling

Theorem. If $a > 0$, then $\mathcal{L}(u(ax))(\omega) = \frac{1}{a}\mathcal{L}(u(t))\left(\frac{s}{a}\right)$.

Notice that we only do this for $a > 0$ (why?).

Proof. We see that

$$\mathcal{L}(u(at))(s) = \int_0^\infty u(at)e^{-st}\,dt = /\ y = at\ / = \int_0^\infty u(y)e^{-sy/a}\,\frac{dy}{a}$$
$$= \frac{1}{a}\int_0^\infty u(y)e^{-(s/a)y}\,dy = \frac{1}{a}\,\mathcal{L}\,u\left(\frac{s}{a}\right).$$

s-shift

Theorem. Suppose that $a \in \mathbf{C}$ is constant. Then $\mathcal{L}(e^{at}u(t))(s) = (\mathcal{L}(u(t)))(s-a)$.

Proof. We note that

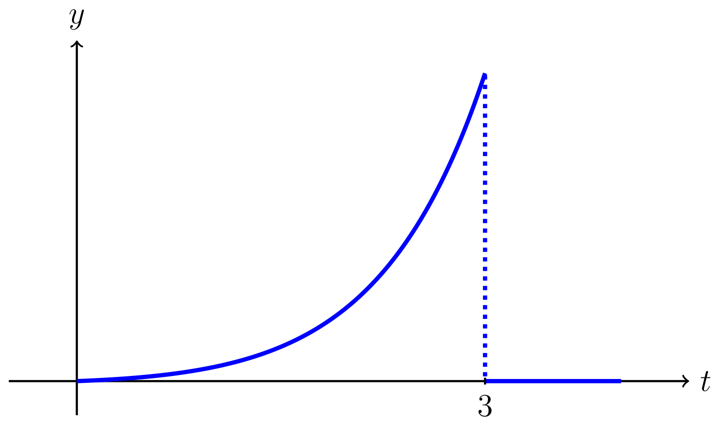
$$\mathcal{L}(e^{at}u(t))(s) = \int_0^\infty u(t)e^{at}e^{-st}\,dt = \int_0^\infty u(t)e^{-(s-a)t}\,dt = (\mathcal{L}\,u)(s-a),$$

which completes the proof.

Example

Find the Laplace transform of $u(t) = e^{2t}tH(3-t)$.

Solution. This is some type of exponential function that's cut off at $t = 3$.



We could plug this into the formula and just do the integration, but we can also apply the rules we now know. So, first observe that

$$v(t) = tH(3-t) = t(H(t) - H(t-3)) = t(H(t) - H(t-3)H(t-3)),$$

so

$$\mathcal{L}(tH(3-t))(s) = -\frac{d}{ds}\left(\mathcal{L}(H(t))(s) - e^{-3s}\,\mathcal{L}(H(t))(s)\right) = -\frac{d}{ds}\left(\frac{1}{s}\left(1-e^{-3s}\right)\right)$$
$$= \frac{1}{s^2}\left(1-e^{-3s}\right) - \frac{3e^{-3s}}{s}.$$

Then

$$\mathcal{L}\,u(s) = \mathcal{L}\,v(s-2) = \frac{1-e^{-3(s-2)}}{(s-2)^2} - \frac{3e^{-3(s-2)}}{s-2} = \frac{1+e^{6-3s}(5-3s)}{(s-2)^2}.$$

9.4.1 Differentiation

One of the major uses for the Laplace transform is how it handles derivatives.

Differentiation

Theorem. Let $u \in X_a$ be continuous such that u' is piecewise continuous. Then

$$\mathcal{L}(u'(t))(s) = s\,\mathcal{L}(u(t))(s) - u(0), \quad \operatorname{Re} s > a.$$

Proof. Let $L > 0$ and let

$$x_0 = 0 < x_1 < x_2 < x_3 < \ldots < x_n = L$$

be the points of discontinuity for u' on $[0, L]$. For $k = 0, 1, 2, \ldots, n-1$, we have

$$\begin{aligned}\int_{x_k}^{x_{k+1}} u'(t)e^{-st}\,dt &= \big[u(t)e^{-st}\big]_{t=x_k}^{t=x_{k+1}} + s\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= u(x_{k+1})e^{-sx_{k+1}} - u(x_k)e^{-sx_k} + s\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt,\end{aligned}$$

so

$$\begin{aligned}\int_0^L u'(t)e^{-st}\,dt &= \sum_{k=0}^{n-1}\big(u(x_{k+1})e^{-sx_{k+1}} - u(x_k)e^{-sx_k}\big) + s\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= \big/ \text{ telescoping sum } \big/ = u(L)e^{-sL} - u(0) + s\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} u(t)e^{-st}\,dt \\ &= u(L)e^{-sL} - u(0) + s\int_0^L u(t)e^{-st}\,dt \to s\,\mathcal{L}\,u(s) - u(0),\end{aligned}$$

since $u(L)e^{-sL} \to 0$ as $L \to \infty$. The last part is clear due to the fact that

$$\big|u(L)e^{-sL}\big| \le Ke^{aL}e^{-L\operatorname{Re} s} \to 0,$$

as $L \to \infty$ ($\operatorname{Re} s > a$). □

Example

Use the Laplace transform to find a solution to $y'(t) + 3y(t) = 0$, $t > 0$, such that $y(0) = 2$.

Solution. Taking the Laplace transform, we find that

$$sY(s) - y(0) + 3Y(s) = 0 \quad\Leftrightarrow\quad Y(s)(s+3) = 2 \quad\Leftrightarrow\quad Y(s) = \frac{2}{s+3},$$

if $\operatorname{Re} s > -3$. We know that $\mathcal{L}(e^{3t}) = 1/(s+3)$, so $y(t) = 2e^{3t}$ is a possible solution. Direct verification shows that this solves the equation in question.

If the function $u \in X_a$ has higher order derivatives (that belong to X_a), one can repeatedly apply the previous theorem to transform higher order derivatives. Indeed, if $u^{(n)}$ is piecewise continuous, then this will hold for all derivatives $u^{(k)}$ as well for $k = 0, 1, 2, \ldots, n$. Thus we obtain the following result.

Higher order derivatives

Corollary. Let $u \in X_a$ be a continuous function such that $u^{(n)}$ is piecewise continuous and $u', u'', \ldots, u^{(n-1)} \in X_a$. Then

$$\begin{aligned}\mathcal{L}(u^{(n)})(s) &= s^n \mathcal{L}\, u(s) - \sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(0) \\ &= s^n \mathcal{L}\, u(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - su^{(n-2)}(0) - u^{(n-1)}(0), \quad \operatorname{Re} s > a.\end{aligned}$$

Example

Find a solution to $y''(t) - 4y(t) = 4e^{2t}$, $t > 0$, with $y(0) = 1$ and $y'(0) = 0$.

Solution. Taking the Laplace transform of both sides in the equality, we find that

$$\begin{aligned}s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{4}{s-2} \quad &\Leftrightarrow \quad Y(s)(s^2-4) = s + \frac{4}{s-2} \\ &\Leftrightarrow \quad Y(s) = \frac{s}{s^2-4} + \frac{4}{(s^2-4)(s-2)}.\end{aligned}$$

Using partial fractions, we find that this expression is equal to

$$\frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right) + \frac{1/4}{s-2} + \frac{1}{(s-2)^2} - \frac{1/4}{s+2}.$$

We see that

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2} \qquad \text{and} \qquad \mathcal{L}(e^{-2t}) = \frac{1}{s+2}.$$

Next we note that

$$\frac{1}{(s-2)^2} = -\frac{d}{ds}\left(\frac{1}{s-2}\right)$$

and by the time multiplication theorem,

$$\mathcal{L}(te^{2t}) = -\frac{d}{ds}\left(\frac{1}{s-2}\right).$$

Hence

$$\mathcal{L}\left(\frac{1}{4}e^{2t} + te^{2t} + \frac{3}{4}e^{-2t}\right) = Y(s),$$

so

$$y(t) = \frac{1}{4}e^{2t} + te^{2t} + \frac{3}{4}e^{-2t}$$

for $t > 0$. Is this the solution?

Example

Note that $\mathcal{L}(\cosh(t)) = \dfrac{s}{s^2-1}$ and $\mathcal{L}(\sinh(t)) = \dfrac{1}{s^2-1}$.

Chapter 10

Convolution and Inversion

"Don't disturb my Friend. He's dead tired."
—John Matrix

10.1 Convolution

In the case when we have two functions $u, v\colon [0,\infty[\to \mathbf{C}$, the convolution gets slightly easier to handle.

Convolution

Definition. The convolution $u*v\colon [0,\infty[\to \mathbf{C}$ of $u\colon [0,\infty[\to \mathbf{C}$ and $v\colon [0,\infty[\to \mathbf{C}$ is defined by

$$(u*v)(t) = \int_0^t u(y)v(t-y)\,dy, \quad 0 \le t < \infty,$$

whenever this integral exists.

Theorem. If $u, v\colon [0,\infty[\to \mathbf{C}$ belong to X_a, then $u*v \in X_b$ for every $b > a$ and

$$|u*v(t)| \le Kte^{at}, \quad t > 0.$$

Furthermore,

$$\mathcal{L}(u*v)(s) = \mathcal{L}(u)(s)\,\mathcal{L}(v)(s), \quad \operatorname{Re} s > a.$$

Proof. By definition, $|u(t)| \le K_1 e^{at}$ and $|v(t)| \le K_2 e^{at}$, so

$$\begin{aligned} |u*v(t)| &= \left|\int_0^t u(\tau)v(t-\tau)\,d\tau\right| \le \text{/ monotonicity /} \le \int_0^t |u(\tau)||v(t-\tau)|\,d\tau \\ &\le \int_0^t Ke^{a\tau}e^{a(t-\tau)}\,d\tau = K_1K_2te^{at}, \end{aligned}$$

and since $\lim_{t\to\infty} te^{-\delta t} = 0$ for any $\delta > 0$, it follows that $|u*v(t)| \le Ke^{bt}$ for every $b > a$.

So the convolution of u and v is defined and belongs to X_b. Taking the Laplace transform, we observe that

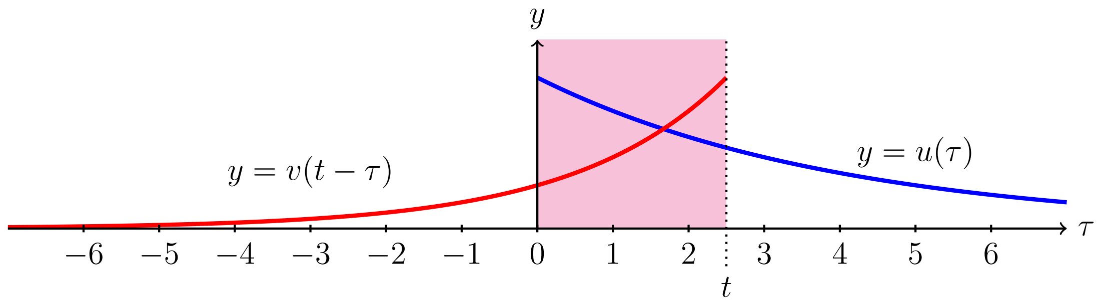
$$\begin{aligned}
\mathcal{L}(u*v)(s) &= \int_0^\infty e^{-st}\int_0^t u(\tau)v(t-\tau)\,d\tau\,dt = \int_0^\infty e^{-s(\tau+(t-\tau))}\int_0^t u(\tau)v(t-\tau)\,d\tau\,dt \\
&= /\text{ Fubini }/ = \int_0^\infty u(\tau)e^{-s\tau}\int_\tau^\infty v(t-\tau)e^{-s(t-\tau)}\,dt\,d\tau \\
&= /\; t = y+\tau \;/ = \int_0^\infty u(\tau)e^{-s\tau}\int_0^\infty v(y)e^{-sy}\,dy\,d\tau \\
&= \mathcal{L}\,v(s)\int_0^\infty u(\tau)e^{-s\tau}d\tau = \mathcal{L}\,v(s)\,\mathcal{L}\,u(s).
\end{aligned}$$

□

Example

Let $u(t) = e^{-t}$ for $t \geq 0$ and $v(t) = e^{-2t}$ for $t \geq 0$. Find $u*v$ and $\mathcal{L}(u*v)(s)$.

Solution. Method 1: direct calculation. First, let's draw the graphs and then mirror the one for v.



Since both u and v are assumed to be zero for negative arguments, the situation is a bit easier than the general convolution we saw when dealing with the Fourier transform. It's only for arguments between zero and t that we obtain something non-zero. Therefore,

$$\begin{aligned}
\int_0^t u(\tau)v(t-\tau)\,d\tau &= \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau = \int_0^t e^{-2t+\tau}\,d\tau = e^{-2t}\int_0^t e^\tau\,d\tau \\
&= e^{-2t}\left[e^\tau\right]_0^t = e^{-2t}(e^t-1) = e^{-t}-e^{-2t},
\end{aligned}$$

so

$$\begin{aligned}
\mathcal{L}(u*v)(s) &= \mathcal{L}(e^{-t}-e^{-2t}) = \mathcal{L}(e^{-t}) - \mathcal{L}(e^{-2t}) = \frac{1}{s+1}-\frac{1}{s+2} = \frac{s+2-(s+1)}{(s+1)(s+2)} \\
&= \frac{1}{(s+1)(s+2)}.
\end{aligned}$$

Method 2: Use the convolution theorem. We find that $\mathcal{L}\,u(s) = \dfrac{1}{s+1}$ and $\mathcal{L}\,v(s) = \dfrac{1}{s+2}$, so

$$\mathcal{L}(u*v)(s) = \mathcal{L}\,u(s)\,\mathcal{L}\,v(s) = \frac{1}{(s+1)(s+2)}.$$

10.2 Periodic Functions

Suppose that there exists some $T > 0$ such that $u(t+T) = u(t)$ for every $t \geq 0$. Formally taking the Laplace transform of u, we find that

$$\begin{aligned}
\mathcal{L}\,u(s) &= \int_0^\infty u(t)e^{-st}\,dt = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} u(t)e^{-st}\,dt \\
&= /\ \tau = t - kT\ / = \sum_{k=0}^\infty \int_0^T u(\tau + kT)e^{-s(\tau+kT)}\,d\tau \\
&= \sum_{k=0}^\infty e^{-skT} \int_0^T u(\tau)e^{-s\tau}\,d\tau = \left(\sum_{k=0}^\infty e^{-skT}\right) \int_0^T u(\tau)e^{-s\tau}\,d\tau \\
&= \frac{1}{1-e^{-sT}} \int_0^T u(\tau)e^{-s\tau}\,d\tau,
\end{aligned}$$

where we used the fact that u is periodic and calculated the resulting geometric series.

Example

Let $u(t) = t$, $0 \leq t < 1$, and $u(t+1) = u(t)$ for $t \geq 0$. Find $\mathcal{L}\,u(s)$.

Solution. Since u is periodic with $T = 1$, we find that

$$\begin{aligned}
\mathcal{L}\,u(s) &= \frac{1}{1-e^{-s}} \int_0^T \tau e^{-s\tau}\,d\tau = \frac{1}{1-e^{-s}} \left(\left[\frac{\tau e^{-s\tau}}{-s}\right]_0^1 + \frac{1}{s}\int_0^T e^{-s\tau}\,d\tau \right) \\
&= \frac{1}{1-e^{-s}} \left(\frac{e^{-s}}{-s} + \frac{1}{s}\left[\frac{e^{-s\tau}}{-s}\right]_0^1 \right) = \frac{1}{1-e^{-s}} \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right) \\
&= \frac{e^s}{e^s - 1} \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right) = \frac{e^s - (s+1)}{s^2(e^s - 1)}.
\end{aligned}$$

Example

Using the periodicity, find the Laplace transform of $u(t) = e^{it}$.

Solution. Since u has period 2π, we find that

$$\begin{aligned}
\mathcal{L}\,u(s) &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{i\tau}e^{-s\tau}\,d\tau = \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{(i-s)\tau}}{i-s}\right]_0^{2\pi} = \frac{1}{1-e^{-2\pi s}} \cdot \frac{e^{2\pi(i-s)} - 1}{i-s} \\
&= \frac{1}{1-e^{-2\pi s}} \cdot \frac{e^{-2\pi s} - 1}{i-s} = \frac{1}{s-i},
\end{aligned}$$

which is presicely what the transform of e^{at} was derived to be (with $a = i$).

10.3 Inversion of the Laplace Transform

Similar to the case with the Fourier transform, there's a formula for the inversion of the Laplace transform as well (which implies certain uniqueness results). We will not use this integral explicitly, but rather use tables to find the correct inverse. However, the fact that we have an inversion result means that we know certain uniqueness properties of the Laplace transform. This is an important fact also when using tables.

Laplace inversion formula

Theorem. If $u \in X_a$ has right- and lefthand limits at a point $t > 0$, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega) e^{\sigma t} e^{i\omega t}\, d\omega = \frac{u(t^+) + u(t^-)}{2},$$

where the vertical line $\operatorname{Re} z = \sigma$ is contained in the region of convergence of $\mathcal{L}\, u(s)$ ($\sigma > a$ is sufficient).

Proof. Since $e^{-\sigma t}$ is continuous, it is clear that $v(t) = H(t)u(t)e^{-\sigma t}$ has left- and righthand limits at all $t > 0$ and belongs to $G(\mathbf{R})$. The fact that $\mathcal{L}\, u(\sigma + i\omega) = \mathcal{F}(e^{-\sigma t} u(t) H(t))(\omega)$ enables us to use Fourier inversion, obtaining that

$$\begin{aligned}
\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega) e^{(\sigma+i\omega)t}\, d\omega &= e^{\sigma t} \lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} (\mathcal{F}\, v)(\omega) e^{i\omega t}\, d\omega \\
&= e^{\sigma t} \frac{v(t^+) + v(t^-)}{2} = e^{\sigma t} \frac{e^{-\sigma t} u(t^+) + e^{-\sigma t} u(t^-)}{2} \\
&= \frac{u(t^+) + u(t^-)}{2},
\end{aligned}$$

which is precisely what was stated in the theorem. □

Similarly with the Fourier transform, a consequence of this result is the following uniqueness theorem.

Uniqueness

Corollary. Suppose that $\mathcal{L}\, u(s)$ and $\mathcal{L}\, v(s)$ are convergent for $\operatorname{Re} s > a$, where $a > 0$, meaning that $u, v \in X_a$. If $\mathcal{L}\, u(s) = \mathcal{L}\, v(s)$ on some vertical line $\operatorname{Re} s = \sigma$, then $u(t) = v(t)$ for all t where u and v are continuous.

Note that we could employ the Fourier result from Lecture 6 as well if $D^{\pm}u(t)$ exists, yielding the same results.

So what use is this in practice? Well, a lot actually. Even if it mostly happens implicitly. Consider the differential equations we've been working with. We solve the equation in the Laplace domain, then find something that gives this Laplace transform and boom. We're done. Or? Well, if we have a uniqueness result, that would be the case (provided that your solution satisfies the required properties). Let's take a look at an example.

Example

Solve the equation

$$y'(t)+y(t)=\begin{cases}\cos t, & 0\le t\le \frac{\pi}{2},\\ 1-\sin t, & t>\frac{\pi}{2},\end{cases}$$

if $y(0)=7$.

Solution. Assume that $y\in X_a$. This is important. We will only find solutions that are bounded by some exponential function. Next we reformulate the right-hand side as

$$\begin{aligned}\left(1-H\left(t-\frac{\pi}{2}\right)\right)\cos t+H\left(t-\frac{\pi}{2}\right)(1-\sin t)&=\cos t+H\left(t-\frac{\pi}{2}\right)(1-\sin t-\cos t)\\ &=\cos t+H\left(t-\frac{\pi}{2}\right)\left(1-\sin\left(t-\frac{\pi}{2}+\frac{\pi}{2}\right)-\cos\left(t-\frac{\pi}{2}+\frac{\pi}{2}\right)\right)\\ &=\cos t+H\left(t-\frac{\pi}{2}\right)\left(1-\cos\left(t-\frac{\pi}{2}\right)+\sin\left(t-\frac{\pi}{2}\right)\right).\end{aligned}$$

The reason for this is that we can use the fact that $\mathcal{L}(u(t-t_0)H(t-t_0))=e^{-st_0}\,\mathcal{L}\,u(s)$. Hence the equation has the Laplace transform

$$sY(s)-7+Y(s)=\frac{s}{s^2+1}+e^{-\pi s/2}\left(\frac{1}{s}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right),$$

so if $\operatorname{Re}s>0$,

$$\begin{aligned}Y(s)&=\frac{7}{s+1}+\frac{s}{(s+1)(s^2+1)}+e^{-\pi s/2}\left(\frac{1}{s(s+1)}+\frac{1-s}{(s+1)(s^2+1)}\right)\\ &=\frac{7}{s+1}+\frac{1}{2}\left(\frac{s}{s^2+1}+\frac{1}{s^2+1}-\frac{1}{s+1}\right)+e^{-s\pi/2}\left(\frac{1}{s}-\frac{1}{s+1}+\frac{1}{s+1}-\frac{s}{s^2+1}\right).\end{aligned}$$

We now observe that

$$\begin{aligned}Y(s)&=\frac{13}{2}\mathcal{L}(e^{-t})+\frac{1}{2}\left(\mathcal{L}(\cos t)+\mathcal{L}(\sin t)\right)+e^{-s\pi/2}\left(\mathcal{L}(H(t))-\mathcal{L}(\cos t)\right)\\ &=\mathcal{L}\left(\frac{13}{2}e^{-t}+\frac{1}{2}\cos t+\frac{1}{2}\sin t+H\left(t-\frac{\pi}{2}\right)\left(1-\cos\left(t-\frac{\pi}{2}\right)\right)\right)\end{aligned}$$

so by uniqueness we claim that

$$\begin{aligned}y(t)&=\frac{13}{2}e^{-t}+\frac{1}{2}\cos t+\frac{1}{2}\sin t+H\left(t-\frac{\pi}{2}\right)\left(1-\cos\left(t-\frac{\pi}{2}\right)\right)\\ &=\frac{13}{2}e^{-t}+\frac{1}{2}\cos t+\frac{1}{2}\sin t+H\left(t-\frac{\pi}{2}\right)(1-\sin t)\\ &=\begin{cases}\frac{13}{2}e^{-t}+\frac{1}{2}\cos t+\frac{1}{2}\sin t, & 0<t<\frac{\pi}{2},\\ \frac{13}{2}e^{-t}+1+\frac{1}{2}\cos t-\frac{1}{2}\sin t, & t\ge\frac{\pi}{2},\end{cases}\end{aligned}$$

which is OK since $y\in X_a$ with $a>0$.

Consider $y' - 2ty = 0$, $y(0) = 1$. Using an integrating factor, we know that $y(t) = e^{t^2}$. *Wrongly* assuming that the solution belongs to X_a for some $a > 0$, we would find that (ignoring any issues with the complex variable),

$$sY(s) - 1 + 2Y'(s) = 0 \quad \Leftrightarrow \quad Y(s) = Ce^{-s^2/4} + se^{-s^2/4}.$$

This is *NOT* the Laplace transform of e^{t^2} (for any constant C). Nope. Nein. Niet. Be very careful when using the uniqueness result!

10.4 Limit Results

Final value theorem

Theorem. Suppose that $u\colon [0,\infty[\to \mathbf{C}$ is a bounded function and that $u(t) \to A$ as $t \to \infty$. Then $A = \lim_{\mathbf{R}\ni s\to 0^+} s\,\mathcal{L}\,u(s)$.

Proof. Let $s \geq 0$ (so real). We use the fact that u is bounded to obtain uniform convergence. The Laplace transform of u exists and

$$s\,\mathcal{L}\,u(s) = \int_0^\infty su(t)e^{-st}\,dt = /\ y = st\ / = \int_0^\infty u\left(\frac{y}{s}\right)e^{-y}\,dy.$$

Since $|u(y/s)|e^{-y} \leq Ce^{-y}$ for some constant C (remember that u is bounded), it is clear that

$$\lim_{s\to 0^+}\int_0^\infty u\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty \lim_{s\to 0^+} u\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty Ae^{-y}\,dy = A\left[-e^{-y}\right]_0^\infty = A.$$

Initial value theorem

Theorem. Suppose that $u\colon [0,\infty[\to \mathbf{C}$ belongs to X_b and that $u(t) \to a$ as $t \to 0^+$. Then $a = \lim_{\mathbf{R}\ni s\to\infty} s\,\mathcal{L}\,u(s)$.

Proof. Since $u \in X_b$, we know that there exists $C > 0$ such that $|u(t)| \leq Ce^{bt}$. Therefore, let $v(t) = e^{-bt}u(t)$. Let $s \geq 0$ (so real). Then $v(0^+) = u(0^+)$ so we might be able to work with v instead. Indeed, the Laplace transform of v exists and

$$s\,\mathcal{L}\,v(s) = \int_0^\infty sv(t)e^{-st}\,dt = /\ y = st\ / = \int_0^\infty v\left(\frac{y}{s}\right)e^{-y}\,dy.$$

Since $|v(y/s)|e^{-y} \leq Ce^{-y}$ for some constant C (this time $v(t) = u(t)e^{-bt}$ is bounded), it is clear that

$$\lim_{s\to\infty}\int_0^\infty v\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty \lim_{s\to\infty} v\left(\frac{y}{s}\right)e^{-y}\,dy = \int_0^\infty a\,e^{-y}\,dy = a\left[-e^{-y}\right]_0^\infty = a.$$

Note now that

$$s\,\mathcal{L}\,v(s) = s\,\mathcal{L}(e^{-bt}u(t))(s) = s\,\mathcal{L}\,u(s-b) = (s-b)\,\mathcal{L}\,u(s-b) + b\,\mathcal{L}\,u(s-b),$$

so since $\mathcal{L}\,u(s) \to 0$ as $s \to \infty$, we obtain that

$$\lim_{s\to\infty} s\,\mathcal{L}\,v(s) = \lim_{s\to\infty} (s+b)\,\mathcal{L}\,u(s+b) = \lim_{s\to\infty} s\,\mathcal{L}\,u(s),$$

which proves that $\lim_{s\to\infty} s\,\mathcal{L}\,u(s) = a$. □

We assume that the limits exist!
In the previous two theorems, we *assume* that the limits exists for the result to hold. It can be the case that the limit in the Laplace domain exists (and seems reasonable) but that the limit in the time domain does *not* exist. This can obviously lead to erroneous deductions.

10.5 More Examples

10.5.1 Convolution Equations

Example

Solve the equations

$$\int_0^t \cos(t-\tau)u(\tau)\,d\tau = f(t), \quad t \geq 0,$$

where

$$\text{(a) } f(t) = t\sin t \qquad \text{(b) } f(t) = 1.$$

Solution. The integral in question is a convolution of $u(t)$ with $\cos t$. Assuming that $u \in X_a$ for some $a > 0$, we take the Laplace transform of both sides in the equality to find that

$$\frac{s}{s^2+1}\,U(s) = \mathcal{L}\,f(s).$$

If $f(t) = t\sin t$, we obtain

$$\mathcal{L}\,f(s) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2},$$

so

$$U(s) = \frac{2}{s^2+1}.$$

Since $\mathcal{L}(2\sin(t))(s) = U(s)$, the uniqueness result proves that $u(t) = 2\sin t$ is the only solution in X_a.
However, if $f(t) = 1$, we find that

$$\frac{s}{s^2+1}\,U(s) = \frac{1}{s} \quad \Leftrightarrow \quad U(s) = \frac{s^2+1}{s^2} = 1 + \frac{1}{s^2}.$$

Now $\mathcal{L}(t) = s^{-2}$, but what would have the Laplace transform 1? It turns out that there's no such *function* (recall that $\mathcal{L}\,u(s) \to 0$ as $|s| \to \infty$ if $u \in X_a$). We can't find a solution in this case. Plugging the equation into some algebra system might give you the answer $u(t) = t + \delta(t)$, whatever that might mean...

10.5.2 Power Series

Since it is allowed to integrate a power series termwise (if we're inside the radius of convergence), we can sometimes find the Laplace transform for a function by using a power series representation. It is straight forward to prove that

$$\mathcal{L}(t^m) = \frac{m!}{s^{m+1}}, \quad \operatorname{Re} s > 0,$$

by either using direct calculation and partial integration or by writing $t^m H(t)$ and using the time multiplication theorem. So if we integrate termwise, we can take the Laplace transform of t^m and then calculate the series.

Example

Find the Laplace transform of $u(t) = \operatorname{sinc}(t) = \dfrac{\sin t}{t}$, $t > 0$.

Solution. We have

$$\operatorname{sinc}(t) = \frac{1}{t}\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k+1)!}, \quad t \in \mathbf{R}.$$

Hence

$$\mathcal{L}(\operatorname{sinc}(t)) = \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{L}(t^{2k})(s)}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{s^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{s^{2k+1}(2k+1)},$$

which is the Maclaurin series for $\arctan\left(\dfrac{1}{s}\right)$. Therefore we have just proved that

$$\mathcal{L}(\operatorname{sinc}(t)) = \arctan\left(\frac{1}{s}\right), \quad \operatorname{Re} s > 1$$

where we leave the difficulties of interpreting this for $s \in \mathbf{C}$ to some other course.

10.5.3 Bessel Functions

A **Bessel function** of order ν solves Bessel's differential equation:

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0.$$

These are usually denoted by $J_\nu(x)$.

Example

Find the Laplace transform of the solution $J_0(x)$ for $x \geq 0$ such that $J_0(1) = 1$.

Solution. Letting $\nu = 0$ and assuming that $x > 0$, we find that

$$xy''(x) + y'(x) + xy(x) = 0.$$

Assuming that $y, y' \in X_a$ for some $a > 0$, we can take the Laplace transform:

$$\begin{aligned} -\frac{d}{ds}\mathcal{L}(y'')(s) + sY(s) - y(0) - \frac{d}{ds}Y(s) = 0 \\ \Leftrightarrow \quad & -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) + sY(s) - y(0) - Y'(s) = 0 \\ \Leftrightarrow \quad & -2sY(s) - s^2Y'(s) + y(0) + sY(s) - y(0) - Y'(s) = 0 \\ \Leftrightarrow \quad & 0 = sY(s) + (s^2+1)Y'(s) \quad \Leftrightarrow \quad Y'(s) + \frac{s}{s^2+1}Y(s) = 0. \end{aligned}$$

Now assume for a moment that s is real. Then we can multiply with the integrating factor

$$\exp\left(\frac{1}{2}\ln(s^2+1)\right) = \left(1+s^2\right)^{1/2}$$

so that

$$Y'(s) + \frac{s}{s^2+1}Y(s) = 0 \quad \Leftrightarrow \quad \frac{d}{ds}\left(Y(s)\left(1+s^2\right)^{1/2}\right) = 0 \quad \Leftrightarrow \quad Y(s) = \frac{C}{(1+s^2)^{1/2}}.$$

To find the value of C, consider the limit theorem from above (assuming that y is continuous):

$$1 = y(0) = \lim_{\mathbf{R}\ni s\to\infty} sY(s) = \lim_{\mathbf{R}\ni s\to\infty} C\frac{s}{(1+s^2)^{1/2}} = C.$$

Hence

$$\mathcal{L}\, J_0(s) = \frac{1}{(1+s^2)^{1/2}},$$

leaving it to a different course how to define this for $s \in \mathbf{C}$.

10.5.4 Linear Systems of Differential Equations

Suppose that we want to solve, for $t > 0$,

$$\begin{cases} x_1'(t) = 4x_1(t) - 2x_2(t) + e^t, \\ x_2'(t) = 3x_1(t) - 3x_2(t) + e^t, \end{cases}$$

where $x_1(0) = 2/3$ and $x_2(0) = -2$. Taking the Laplace transform, we obtain that

$$\begin{cases} sX_1(s) - \dfrac{2}{3} = 4X_1(s) - 2X_2(s) + \dfrac{1}{s-1}, \\ sX_2(s) + 2 = 3X_1(s) - 3X_2(s) + \dfrac{1}{s-1}, \end{cases} \quad \Leftrightarrow \quad \begin{cases} (s-4)X_1(s) + 2X_2(s) = \dfrac{2}{3} + \dfrac{1}{s-1}, \\ -3X_1(s) + (s+3)X_2(s) = -2 + \dfrac{1}{s-1}. \end{cases}$$

Let

$$A(s) = \begin{pmatrix} s-4 & 2 \\ -3 & s+3 \end{pmatrix}, \qquad X(s) = \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \quad \text{and} \quad b(s) = \begin{pmatrix} 2/3 + 1/(s-1) \\ -2 + 1/(s-1) \end{pmatrix}$$

so $A(s)X(s) = b(s)$. Assuming that $\det A \neq 0$, we have

$$\begin{aligned} X(s) = A^{-1}b(s) &= \frac{1}{(s-4)(s+3)+6} \begin{pmatrix} s+3 & -2 \\ 3 & s-4 \end{pmatrix} \begin{pmatrix} 2/3 + 1/(s-1) \\ -2 + 1/(s-1) \end{pmatrix} \\ &= \frac{1}{(s-3)(s+2)} \begin{pmatrix} \dfrac{2s^2+19s-15}{3(s+2)} \\ -2s+11 \end{pmatrix}. \end{aligned}$$

To find something that transforms to this vector, we use a partial fractions decomposition to see that

$$X_1(s) = \frac{-1/3}{s-1} + \frac{-1}{s+2} + \frac{2}{s-3} \quad \text{and} \quad X_2(s) = \frac{-3}{s+2} + \frac{1}{s-3}.$$

Noting that

$$\mathcal{L}\left(-\frac{1}{3}e^t - e^{-2t} + 2e^{3t}\right) = X_1(s) \quad \text{and} \quad \mathcal{L}\left(-3e^{-2t} + 2e^{3t}\right) = X_2(s),$$

we claim that the unique solution in X_a to the problem is given by

$$\begin{cases} x_1(t) = -\dfrac{1}{3}e^t - e^{-2t} + 2e^{3t}, \\ x_2(t) = -3e^{-2t} + 2e^{3t}. \end{cases}$$

Plugging this into the differential equation also shows that this actually is a solution.

Chapter 11

The Unilateral Z-transform

"Go ahead. Won't show on this shirt"
—Ben Richards

11.1 Complex Power Series

Let $u_0, u_1, u_2, \ldots$ be a sequence of complex numbers. We will use the notation $u[k] = u_k$. The square brackets indicate that the *function* $u\colon \mathbf{N} \to \mathbf{C}$ is defined on the natural numbers $\mathbf{N}$ (which includes zero in this setting). The **power series** corresponding to this sequence is defined by

$$\sum_{k=0}^{\infty} u[k]z^k = \lim_{n\to\infty} \sum_{k=0}^{n} u[k]z^k,$$

whenever this limit exists. So it is a sequence of complex numbers and convergence is defined on $\mathbf{C}$ as expected (we've used this implicitly already throughout the course).

Existence

Theorem. A complex power series has a radius of convergence R such that $\sum_{k=0}^{\infty} u[k]z^k$ is absolutely convergent if $|z| < R$ and divergent if $|z| > R$.

The behavior when $|z| = R$ (meaning all points in $\mathbf{C}$ of the form $Re^{i\theta}$) is not known at this point (and we will not delve deeper into this subject in this course).
As usual, we find the region of convergence by Cauchy's root-test (or de'Alemberts ratio test). The root-test states that if

$$\limsup_{k\to\infty} \left|u[k]z^k\right|^{1/k} < 1, \tag{11.1}$$

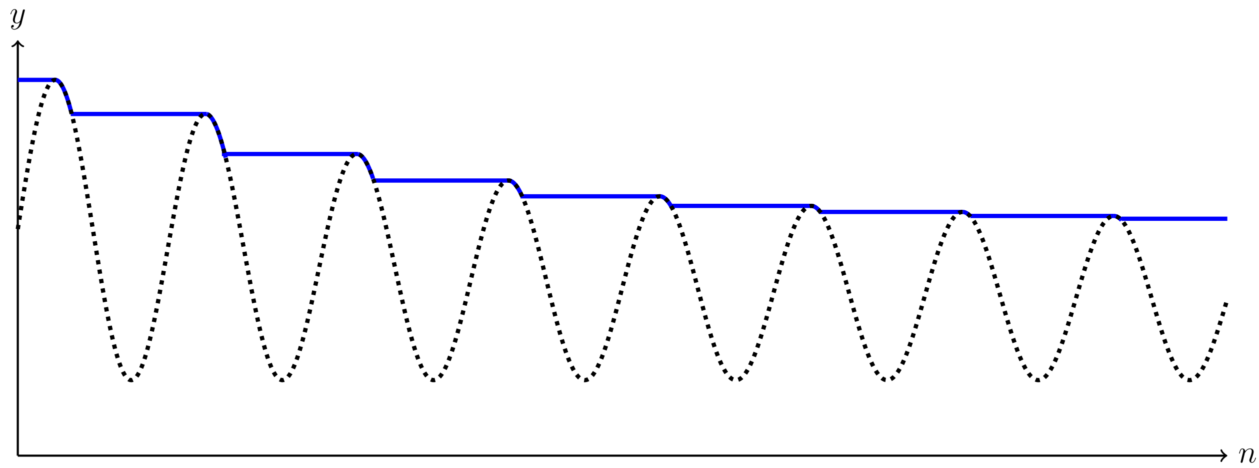
then $\sum_{k=0}^{\infty} u[k]z^k$ is absolutely convergent, and if the limit is > 1, then the series is divergent. In the case when

$$\lim_{k\to\infty} \left|u[k]z^k\right|^{1/k}$$

exists, this limit is equal to the left-hand side of (11.1) above. We actually find the region of convergence by first calculating the limit and then solving for $|z|$. Why involve the weird *limes superior*? Well, this expression always exist, so that's basically what's needed to ensure that we always have a region of convergence (even if the regular limit doesn't exist like in the case of a sum of geometric series' with different quotient). If x_n is a sequence of real numbers, we define $\limsup_{n\to\infty} x_n$ by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right).$$

Note that this expression always exist (although it might be $+\infty$) since it is the limit of a decreasing sequence. As an example, consider the two functions below. The dotted curve is the sequence x_n and the blue curve is $\sup_{m\geq n} x_m$, that is, the smallest upper bound to x_m for $m \geq n$.



Uniqueness

Theorem. If $\sum_{k=0}^{\infty} u[k]z^k = \sum_{k=0}^{\infty} v[k]z^k$ for $|z| < R$, where R is some positive constant, then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$

Basically this means that if two power series converge to the same function in some open disc, then *all* the coefficients must be equal. This is a very useful result.

11.1.1 Uniform Convergence

We've seen in courses previously that you may differentiate (and integrate) power series termwise. The reason for this is basically that they converge uniformly. To see why the convergence is uniform for $|z| \leq r < R$, note the following. Choose some r_0 such that $r < r_0 < R$. Since $\limsup_{k\to\infty} |u[k]z^k|^{1/k} < 1$, there exists some integer $N > 0$ such that

$$k \geq N \quad \Rightarrow \quad |u[k]z^k|^{1/k} \leq \frac{r}{r_0}.$$

So for $k \geq N$ (and $z \neq 0$), we have

$$|u[k]z^k|^{1/k} \leq \frac{r}{r_0} \quad \Leftrightarrow \quad |u[k]z^k| \leq \left(\frac{r}{r_0}\right)^k.$$

Since $r < r_0 < R$, letting $\rho = r/r_0$ we see that $0 < \rho < 1$ and that

$$|u[k]z^k| \le \rho^k, \quad k \ge N.$$

Thus

$$\sum_{k=0}^{\infty} |u[k]z^k| \le \sum_{k=0}^{N-1} |u[k]|r^k + \sum_{k=N}^{\infty} \rho^k < \infty,$$

so by Weierstrass' M-test, the convergence is uniform. So does this mean we can differentiate termwise? Not exactly. However, considering the series of the termwise derivative we see that this also is a power series and that

$$\sum_{k=1}^{\infty} ku[k]z^{k-1} = z^{-1}\sum_{k=0}^{\infty} ku[k]z^k.$$

Note that

$$|ku[k]z^k|^{1/k} = k^{1/k}\,|u[k]z^k|^{1/k}$$

and since $k^{1/k} \to 1$ as $k \to \infty$ it is clear that

$$\limsup_{k\to\infty} |ku[k]z^k|^{1/k} = \limsup_{k\to\infty} |u[k]z^k|^{1/k},$$

so we will obtain the same radius of convergence. Obviously the series of the derivatives also converge uniformly, so yes, we are allowed to differentiate termwise for a power series. Awesome!

11.2 The Unilateral Z transform

The unilateral Z transform

Definition. For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, we define the **Z transform** of u by

$$\mathcal{Z}(u)(z) = \sum_{k=0}^{\infty} u[k]\, z^{-k},$$

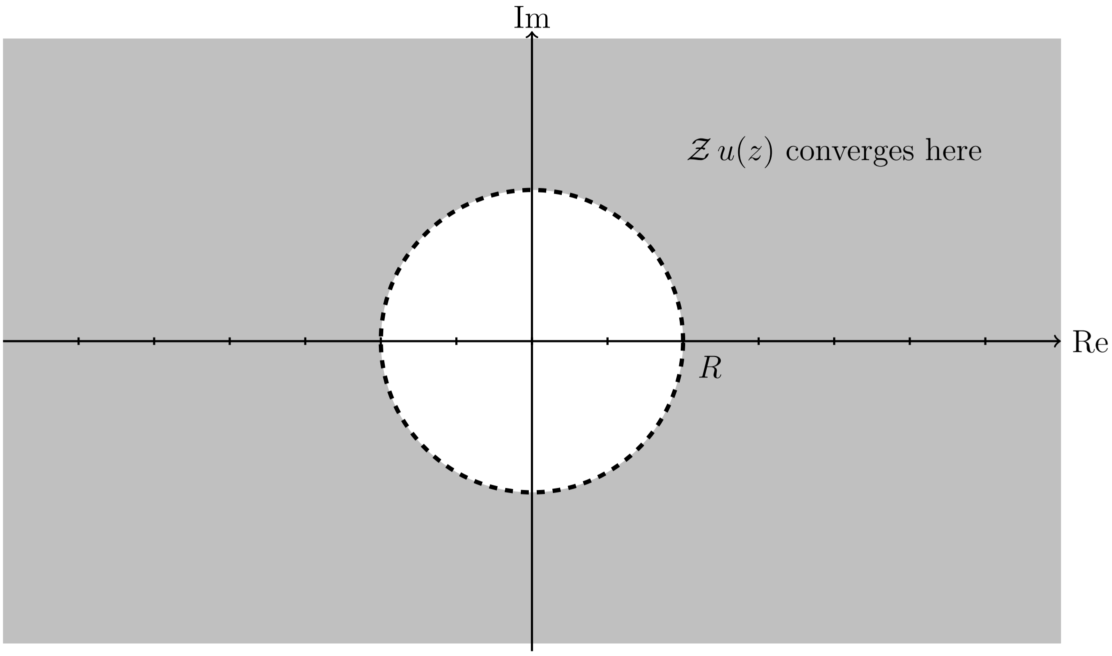
whenever the series is convergent.

We note immediately that this is a power series in the variable z^{-1}. This means that the series has a radius of convergence, but that the series is convergent *outside* a circle with this radius. We have the following result.

Region of convergence

Theorem. For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, the Z transform $\mathcal{Z}\, u(z)$ has a region of convergence defined by the radius of convergence R such that $\mathcal{Z}\, u(z)$ is absolutely (uniformly) convergent for $|z| > R$ and divergent for $|z| < R$. It is possible that $R = 0$.

Proof. This result follows from the existence result for power series by letting $w = z^{-1}$ and considering $\sum_{k=0}^{\infty} u[k]w^k$. □



Example

The geometric series $\sum_{k=0}^{\infty} z^{-k}$ converges if $|z^{-1}| < 1$ and diverges if $|z^{-1}| > 1$. If $|z| > 1$, we have $\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$.

Impulse function and discrete Heaviside

Definition. We define the discrete impulse function $\delta[n]$ by $\delta[n] = 1$ if $n = 0$ and $\delta[n] = 0$ if $n \neq 0$. The discrete Heaviside function $H[n]$ is defined by $H[n] = 1$ if $n \geq 0$ and $H[n] = 0$ if $n < 0$.

Example

Find the Z transform of $u[n] = \delta[n]$.

Solution. Obviously $\mathcal{Z}(\delta[n])(z) = 1$.

Example

Find the Z transform of $u[k] = 1$, $k = 0, 1, 2, \ldots$

Solution. We find that

$$\mathcal{Z}\,u(z) = \sum_{k=0}^{\infty} \frac{1}{z^k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad |z| > 1,$$

since this is a geometric series. Note that this is the Z transform of $H[k]$.

Example

Find the Z transform of $u[k] = k$, $k = 0, 1, 2, \ldots$

Solution. We find that

$$\mathcal{Z}\,u(z) = \sum_{k=0}^{\infty} \frac{k}{z^k} = -z\frac{d}{dz}\sum_{k=0}^{\infty} \frac{1}{z^k} = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}, \quad |z| > 1,$$

since this is the derivative of a geometric series (remember TATA42?).

Example

Find the Z transform for $u[k] = \dfrac{1}{k!}$.

Solution. We identify the coefficients in the Z transform as those of the Maclaurin series for the exponential function, so

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^k = e^{1/z}.$$

11.3 Rules for the Z Transform

Linearity

Theorem. Suppose that $\mathcal{Z}\,u(z)$ and $\mathcal{Z}\,v(z)$ exists for $|z| > R$. Then

$$\mathcal{Z}\big(c_1 u[k] + c_2 v[k]\big)(z) = c_1\,\mathcal{Z}\,u(z) + c_2\,\mathcal{Z}\,v(z), \quad |z| > R.$$

Proof. Let $\mathcal{Z}\,u(z)$ and $\mathcal{Z}\,v(z)$ have the radius of convergence R_u and R_v respectively. By defining $R = \max\{R_u,\ R_v\}$, the linearity follows since the summation is linear when all the sums are convergent. □

Geometric multiplier

Theorem. If $a \neq 0$, then $\mathcal{Z}(a^k u[k])(z) = \mathcal{Z}(u[k])\left(\frac{z}{a}\right)$.

Proof. Taking the Z-transform, we find that

$$\mathcal{Z}(a^k u[k])(z) = \sum_{k=0}^{\infty} a^k u[k] z^{-k} = \sum_{k=0}^{\infty} u[k]\left(\frac{z}{a}\right)^{-k} = \mathcal{Z}\,u\left(\frac{z}{a}\right),$$

under the condition that $|z| > |a|R$ where $R > 0$ is the radius of convergence for $\mathcal{Z}\,u(z)$. □

Example

Show that $\mathcal{Z}(a^k) = \dfrac{z}{z-a}$, $|z| > |a|$.

Solution. Recall that $\mathcal{Z}(H) = \dfrac{z}{z-1}$, so by the previous result we obtain that

$$\mathcal{Z}(a^k) = \mathcal{Z}(a^k H(k)) = \frac{z/a}{z/a - 1} = \frac{z}{z-a}, \qquad |z| > |a|.$$

Example

Find the Z-transforms for $\cos k\alpha$ and $\sin k\alpha$.

Solution. Using Euler's equations, we find that

$$\begin{aligned}\mathcal{Z}\left(\frac{e^{ik\alpha}+e^{-ik\alpha}}{2}\right) &= \frac{1}{2}\left(\frac{z}{z-e^{i\alpha}} + \frac{z}{z-e^{-i\alpha}}\right) = \frac{1}{2}\left(\frac{z(z-e^{-i\alpha}) + z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right) \\ &= \frac{1}{2}\left(\frac{2z^2 - z(e^{-i\alpha}+e^{i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right) = \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1},\end{aligned}$$

since $\mathcal{Z}(a^k) = z/(z-a)$ for $a \in \mathbf{C}$ $(a \neq 0)$. Analogously,

$$\begin{aligned}\mathcal{Z}\left(\frac{e^{ik\alpha}-e^{-ik\alpha}}{2i}\right) &= \frac{1}{2i}\left(\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right) = \frac{1}{2i}\left(\frac{z(z-e^{-i\alpha}) - z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right) \\ &= \frac{1}{2i}\left(\frac{z(e^{i\alpha}-e^{-i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right) = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}.\end{aligned}$$

In both cases we have $|z| > 1$.

Conjugation

Theorem. $\mathcal{Z}(\overline{u[k]})(z) = \overline{\mathcal{Z}(u[k])(\overline{z})}$

Proof. Clearly

$$\mathcal{Z}(\overline{u[k]})(z) = \sum_{k=0}^{\infty} \overline{u[k]} z^{-k} = \sum_{k=0}^{\infty} \overline{u[k]\overline{z}^{-k}} = \overline{\sum_{k=0}^{\infty} u[k]\overline{z}^{-k}} = \overline{\mathcal{Z}(u[k])(\overline{z})}$$

for $|z| > R$. □

11.3.1 Time Shifts

One of the major uses for the Z-transform is its ability to handle delayed signals, meaning expressions of the type $u[k-1]$ etc where we need the value at the previous time step. A special case of this occurs when solving difference equations (we'll see that below).

Time shift

Theorem. For $m > 0$ an integer,

(i) $\mathcal{Z}(u[k+m])(z) = z^m\,\mathcal{Z}(u[k])(z) - \sum_{k=0}^{m-1} u[l]z^{m-k}$,

(ii) $\mathcal{Z}(u[k-m])(z) = z^{-m}\,\mathcal{Z}(u[k])(z) + \sum_{k=-m}^{-1} u[k]z^{-(m+k)}$ (assuming that u is defined for these values) and

(iii) $\mathcal{Z}(u[k-m]H[n-m])(z) = z^{-m}\,\mathcal{Z}(u[k])(z)$.

Proof. We obtain these results by reindexing the series'.

(i)

$$\begin{aligned}\mathcal{Z}(u[k+m])(z) &= \sum_{k=0}^{\infty} u[k+m]z^{-k} = \sum_{k=m}^{\infty} u[k]z^{-(k-m)} = z^m \sum_{k=m}^{\infty} u[k]z^{-k}\\ &= z^m\left(\sum_{k=0}^{\infty} u[k]z^{-k} - \sum_{k=0}^{m-1} u[k]z^{-k}\right) = z^m\,\mathcal{Z}\,u(z) - \sum_{k=0}^{m-1} u[k]z^{m-k}\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{Z}(u[k-m])(z) &= \sum_{k=0}^{\infty} u[k-m]z^{-k} = \sum_{k=-m}^{\infty} u[k]z^{-k-m} = z^{-m} \sum_{k=-m}^{\infty} u[k]z^{-k}\\ &= z^{-m}\left(\sum_{k=-m}^{-1} u[k]z^{-k} + \sum_{k=0}^{\infty} u[k]z^{-k}\right) = z^{-m}\,\mathcal{Z}\,u(z) + \sum_{k=-m}^{-1} u[k]z^{-m-k}\end{aligned}$$

(iii)

$$\begin{aligned}\mathcal{Z}(u[k-m]H[k-m])(z) &= \sum_{k=0}^{\infty} u[k-m]H[k-m]z^{-k} = \sum_{k=-m}^{\infty} u[k]H[k]z^{-k-m}\\ &= z^{-m}\sum_{k=0}^{\infty} u[k]z^{-k} = z^{-m}\,\mathcal{Z}\,u(z).\end{aligned}$$

Difference Equations

One of the major uses for the (unilateral) Z-transform is solving linear difference equations.

Example

Find a solution to $2u[k+1] - u[k] = 1$, $k = 0, 1, 2, \dots$, $u[0] = 2$.

Solution. Taking the Z-transform, we find that for $|z| > 1$,

$$2(zU(z) - zu[0]) - U(z) = \frac{z}{z-1} \quad \Leftrightarrow \quad (2z-1)U(z) = 4z + \frac{z}{z-1}$$
$$\Leftrightarrow \quad U(z) = \frac{4z}{2z-1} - \frac{z}{(z-1)(2z-1)}.$$

Using partial fractions, we obtain

$$U(z) = \frac{4z}{2z-1} + \frac{1}{z}\left(\frac{z}{z-1} - \frac{2z}{2z-1}\right).$$

Thus we expect that

$$u[k] = 2 \cdot 2^{-k} + H[k-1]\left(1 - \left(\frac{1}{2}\right)^{k-1}\right) = 2^{1-k} + H[k-1]\left(1 - 2^{1-k}\right).$$

For $k \geq 1$ we have $2^{1-k} + 1 - 2^{1-k} = 1$ and for $k = 0$ we have $u[0] = 2$. Therefore we obtain that $u[k] = 1$ for $k = 1, 2, 3, \ldots$ and $u[0] = 2$. Verify directly!

Example

Find a solution to $u[k+2] + u[k+1] - 2u[k] = 3\delta[k]$, $k = 0, 1, 2, \ldots$, $u[0] = 0$ and $u[1] = 3$.

Solution. Taking the Z-transform, we find that

$$z^2U(z) - z^2u[0] - zu[1] + zU(z) - zu[0] - 2U(z) = 3 \quad \Leftrightarrow \quad (z^2 + z - 2)U(z) = 3 + 3z$$
$$\Leftrightarrow \quad U(z) = \frac{3+3z}{z^2+z-2},$$

at least if $|z| > 1$. Why? Well, $z^2 + z - 2 = (z+2)(z-1)$ so we have poles at -2 and 1 (zeroes of the polynomial in the denominator). Decomposing by partial fractions and reformulating slightly, we find that

$$U(z) = \frac{1}{z+2} + \frac{2}{z-1} = \frac{1}{z}\frac{z}{z+2} + \frac{1}{z}\frac{2z}{z-1}.$$

Similar to the previous example, we obtain that

$$u[k] = (-2)^{k-1}H(k-1) + 2H[k-1]H[k-1] = \begin{cases} (-2)^{k-1} + 2, & \text{for } k \geq 1, \\ 0, & \text{for } k = 0. \end{cases}$$

Verifying, we see that for $k = 0$:

$$u[0+2] + u[0+1] - 2u[0] = (-2)^1 + 2 + 3 - 2 \cdot 0 = 3,$$

for $k = 1$:

$$u[1+2] + u[1+1] - 2u[1] = (-2)^2 + 2 + (-2)^1 + 2 - 2 \cdot 3 = 0,$$

and for $k \geq 2$:

$$\begin{aligned} u[k+2] + u[k+1] - 2u[k] &= (-2)^{k+1} + 2 + (-2)^k + 2 - 2 \cdot \left((-2)^{k-1} + 2\right) \\ &= (-2)^{k+1} + (-2)^k + (-2)^k = (-2)^k(-2+2) = 0. \end{aligned}$$

11.3.2 Derivatives

Theorem. Suppose that $\mathcal{Z}\,u(z)$ converges for $|z| > R$. Then $\mathcal{Z}(ku[k])(z) = -z\dfrac{d}{dz}\,\mathcal{Z}(u[k])(z)$ for $|z| > R$.

Proof. We see that

$$\begin{aligned}\mathcal{Z}\,u(z) &= \sum_{k=0}^{\infty} u[k]kz^{-k} = -z\sum_{k=0}^{\infty} u[k]\frac{d}{dz}z^{-k} \\ &= /\text{ uniform convergence } / = -z\frac{d}{dz}\sum_{k=0}^{\infty} u[k]z^{-k} = -z\frac{d}{dz}\,\mathcal{Z}\,u(z), \quad |z| > R,\end{aligned}$$

where some care is needed since this is a complex derivative. □

Example

For $a \in \mathbf{C}$, $a \neq 0$, show that for $|z| > |a|$,

$$\mathcal{Z}(a^k)(z) = \frac{z}{z-a}, \qquad \mathcal{Z}(ka^k)(z) = \frac{az}{(z-a)^2} \quad \text{and} \quad \mathcal{Z}(k^2a^k)(z) = \frac{az^2 + a^2z}{(z-a)^3}.$$

Solution. We find that

$$\mathcal{Z}(a^k)(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|.$$

From this it follows that

$$\mathcal{Z}(ka^k)(z) = -z\frac{d}{dz}\,\mathcal{Z}(a^k)(z) = -z\frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}$$

and that

$$\mathcal{Z}(k^2a^k)(z) = -z\frac{d}{dz}\,\mathcal{Z}(ka^k)(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}\,\mathcal{Z}(a^k)(z)\right) = -z\frac{d}{dz}\frac{az}{(z-a)^2} = \frac{a^2z + a^2z}{(z-a)^3}.$$

11.3.3 Binomial Coefficients

Remember that the binomial coefficients were defined by

$$\binom{k}{m} = \frac{k!}{(k-m)!\,m!}, \quad k = 0, 1, 2, \ldots, \; m = 0, 1, 2, \ldots, k.$$

For $k < m$, we let $\binom{k}{m} = 0$ (this might be new).

Theorem. $\mathcal{Z}\left(\binom{k}{m}a^k\right)(z) = \dfrac{a^m z}{(z-a)^{m+1}}$, $m = 0, 1, 2, \ldots$, $|z| > |a|$.

Proof. First, let's consider the Z-transform of $\binom{k}{m}$ for some fixed m:

$$
\begin{aligned}
\mathcal{Z}\binom{k}{m}(z) &= \sum_{k=0}^{\infty}\binom{k}{m}z^{-k} = \frac{1}{m!}\sum_{k=m}^{\infty}\frac{k!}{(k-m)!}z^{-k} \\
&= \frac{1}{m!}\sum_{k=m}^{\infty}k(k-1)(k-2)\cdots(k-m+1)z^{-k} \\
&= \Big/\ w = z^{-k}\ \Big/ = \frac{1}{m!}\sum_{k=m}^{\infty}w^m\frac{d^m}{dw^m}w^k = \frac{w^m}{m!}\frac{d^m}{dw^m}\sum_{k=m}^{\infty}w^k \\
&= \Bigg/\ \frac{d^m}{dw^m}\sum_{k=0}^{m-1}w^k = 0\ \Bigg/ = \frac{w^m}{m!}\frac{d^m}{dw^m}\sum_{k=0}^{\infty}w^k = \frac{w^m}{m!}\frac{d^m}{dw^m}\frac{1}{1-w} \\
&= \frac{w^m}{m!}\frac{m!}{(1-w)^{m+1}} = \frac{w^m}{(1-w)^{m+1}} = \frac{z}{(z-1)^{m+1}}.
\end{aligned}
$$

From this it follows that

$$
\mathcal{Z}\left(\binom{k}{m}a^k\right)(z) = \frac{z/a}{(z/a-1)^{m+1}} = \frac{a^m z}{(z-a)^{m+1}}.
$$

Corollary. $\mathcal{Z}\left(\binom{k+n}{m}a^k\right)(z) = \dfrac{a^{m-n}z^{n+1}}{(z-a)^{m+1}}$, $m = 1, 2, 3, \ldots$, $n = 0, 1, \ldots, m-1$.

Proof. Since $\binom{l}{m} = 0$ for $l < m$, it follows that

$$
\begin{aligned}
\mathcal{Z}\left(\binom{k+n}{m}a^k\right)(z) &= a^{-n}\,\mathcal{Z}\left(\binom{k+n}{m}a^{k+n}\right)(z) = a^{-n}z^n\,\mathcal{Z}\left(\binom{k}{m}a^k\right)(z) \\
&= a^{-n}z^n\frac{a^m z}{(z-a)^{m+1}} = \frac{a^{m-n}z^{n+1}}{(z-a)^{m+1}}.
\end{aligned}
$$

Example

Find a solution to $4u[k+2] - 4u[k+1] + u[k] = 4 \cdot 2^{-k}$, $k = 0, 1, 2, \ldots$, $u[0] = 1$ and $u[1] = 1$.

Solution. Taking the Z-transform of both sides of the equation, we find that for $|z| > 1/2$,

$$\begin{aligned} 4\left(z^2U(z) - z^2u[0] - zu[1]\right) - 4\left(zU(z) - zu[0]\right) + U(z) &= \frac{4z}{z-\frac{1}{2}} \\ \Leftrightarrow \quad (4z^2 - 4z + 1)U(z) &= \frac{4z}{z-\frac{1}{2}} + 4z^2 \\ \Leftrightarrow \quad U(z) = \frac{z}{\left(z-\frac{1}{2}\right)^3} + \frac{z^2}{\left(z-\frac{1}{2}\right)^2} &= 4\frac{2^{-2}z}{\left(z-\frac{1}{2}\right)^3} + \frac{z^2}{\left(z-\frac{1}{2}\right)^2}. \end{aligned}$$

By the previous corollary, we know that

$$\mathcal{Z}\left(\left(\begin{array}{c} k \\ 2 \end{array}\right) a^k\right)(z) = \frac{a^2 z}{(z-a)^3} \quad \text{and} \quad \mathcal{Z}\left(\left(\begin{array}{c} k+1 \\ 1 \end{array}\right) a^k\right)(z) = \frac{z^2}{(z-a)^2},$$

so by linearity we expect that

$$u[k] = 4\left(\begin{array}{c} k \\ 2 \end{array}\right) 2^{-k} + \left(\begin{array}{c} k+1 \\ 1 \end{array}\right) 2^{-k} = \left(2k(k-1) + (k+1)\right) 2^{-k} = \left(2k^2 - k + 1\right) 2^{-k},$$

for $k = 2, 3, 4, \ldots$, solves the equation.
Verifying, we see that for $k = 0$:

$$4u[0+2] - 4u[0+1] + u[0] = 7 - 4 + 1 = 4,$$

for $k = 1$:

$$u[1+2] + u[1+1] - 2u[1] = 8 - 7 + 1 = 2 = 4 \cdot 2^{-1},$$

and for $k \geq 2$:

$$u[k+2] + u[k+1] - 2u[k] = \cdots = 4 \cdot 2^{-k}.$$

Chapter 12

Inversion, Convolution and Bilateral Transforms

"I let him go."
—John Matrix

12.1 Inversion

We first note that the uniqueness of a power series representation (there's only one Maclaurin series) means that we have the following result.

Uniqueness of the Z-transform

Theorem. Suppose that $u[k]$ and $v[k]$ have the same Z-transform, that is, $\mathcal{Z}\,u(z) = \mathcal{Z}\,v(z)$ for all $|z| > R$ for some $R > 0$. Then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$

For a given Z-transform $U(z)$, we typically find $u[k]$ as we did for the Laplace transform, meaning that we need to rewrite $U(z)$ until we can find the components in a table. The uniqueness then proves that the answer is the only possibility. There is an inversion formula as well, that looks like this:

$$u[k] = \frac{1}{2\pi i} \oint_{\gamma} z^{k-1} U(z)\, dz,$$

where γ is a closed curve completely inside the region of converges that loops once around the origin with positive orientation (counter clockwise). Choosing some $r > R$, where R is the radius of convergence of $U(z)$, and letting γ be the closed circle with center at the origin and radius r, i.e., $z = re^{i\theta}$ with $0 \le \theta \le 2\pi$, parametrizing the integral above we obtain that

$$u[k] = \frac{1}{2\pi i} \int_0^{2\pi} r^{k-1} e^{i(k-1)\theta}\, U(re^{i\theta}) r i e^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta}\, U(re^{i\theta})\, d\theta.$$

Why does this work? Since $U(z)$ is a power series, we are allowed to integrate termwise:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta}\, U(re^{i\theta})\, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} r^k e^{ik\theta} \left(\sum_{m=0}^{\infty} u[m] r^{-m} e^{-im\theta} \right) d\theta \\ &= \frac{1}{2\pi} \sum_{m=0}^{\infty} r^{k-m} u[m] \int_0^{2\pi} e^{i(k-m)\theta}\, d\theta = \frac{1}{2\pi} r^0 u[k] \cdot 2\pi = u[k], \end{aligned}$$

since $\displaystyle\int_0^{2\pi} e^{i(k-m)\theta}\, d\theta = 0$ if $k \neq m$.
This result implies the following theorem. Note that we need a condition for the behavior of U "at infinity."

Theorem. Suppose that $U(z)$ is analytic for $|z| > R$, where $R > 0$ is some constant. If $\lim_{|z|\to\infty} U(z) = A$ for some $A \in \mathbf{C}$, then there exists a unique function $u\colon \mathbf{N} \to \mathbf{C}$ such that $\mathcal{Z}\, u(z) = U(z)$ for $|z| > R$.

Example

Suppose that $U(z) = \log(1 + z^{-1})$, $|z| > 1$. Is there some $u[k]$ such that $U(z) = \mathcal{Z}\, u(z)$?

Solution. So $U(z)$ is a bit problematic if you haven't studied complex analysis, but let's ignore that and just work with this formally. Remember that the Maclaurin series is given by

$$\log(1 + z^{-1}) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\, z^{-k}, \quad |z| > 1,$$

so from this it is clear that $u[k] = \dfrac{(-1)^k}{k}$ for $k = 1, 2, 3, \dots$ By uniqueness, this is the only possibility. Note that $U(z) \to 0$ as $|z| \to \infty$.

12.2 Discrete Convolution

Definition. For $u, v\colon \mathbf{Z} \to \mathbf{C}$, we define the **convolution** $u * v$ by

$$u * v[n] = \sum_{k=-\infty}^{\infty} u[k] v[n-k],$$

whenever this series exists.

So an obvious question is when this limit exists.

Theorem. If $u, v \in l^1$, then $u * v \in l^1$.

Proof. We first prove that $u * v$ is absolutely integrable:

$$\sum_{n=-\infty}^{\infty} |u*v[n]| = \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} u[k]v[n-k] \right| \leq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |u[k]||v[n-k]|$$
$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |u[k]||v[n-k]| = \sum_{k=-\infty}^{\infty} |u[k]| \sum_{n=-\infty}^{\infty} |v[n-k]|$$

Note now that

$$\sum_{n=-\infty}^{\infty} |v[n-k]| = /\ m = n-k\ / = \sum_{m=-\infty}^{\infty} |v[m]|,$$

so

$$\sum_{k=-\infty}^{\infty} |u[k]| \sum_{n=-\infty}^{\infty} |v[n-k]| = \left(\sum_{k=-\infty}^{\infty} |u[k]| \right) \left(\sum_{m=-\infty}^{\infty} |v[m]| \right) < \infty.$$

A more compact way of stating this result is that

$$\|u*v\|_{l^1(\mathbf{Z})} \leq \|u\|_{l^1(\mathbf{Z})} \|v\|_{l^1(\mathbf{Z})}.$$

The right-hand side is finite by assumption. □

Did the proof look familiar? It should, go back to lecture 7 and see how we proved the analogous result for the continuous convolution of L^1-functions.
We will soon take a look at the Z-transform of a convolution, and since we're only working with the unilateral Z-transform, we can assume that $u[k] = v[k] = 0$ for $k < 0$. The convolution reduces to

$$u*v[n] = \sum_{k=0}^{n} u[k]v[n-k], \quad n = 0, 1, 2, \ldots$$

In this case, we can relax the conditions a bit and still obtain convergence.

Unilateral convolution

Theorem. If $u, v\colon \mathbf{N} \to \mathbf{C}$ belong to X_a (meaning that $|u[k]| \leq K a^k$ for some $K > 0$ and $a > 0$), then $u * v \in X_b$ for every $b > a$ and

$$|u*v[k]| \leq C\,(k+1)\,a^k, \quad k \geq 0.$$

Furthermore,

$$\mathcal{Z}(u*v)(z) = \mathcal{Z}\,u(z)\,\mathcal{Z}\,v(z), \quad |z| > a.$$

Proof. By definition, $|u[k]| \leq C_1 a^k$ and $|v[k]| \leq C_2 a^k$, so

$$|u*v[n]| = \left| \sum_{k=0}^{n} u[k]v[n-k] \right| \leq /\ \text{monotonicity}\ / \leq \sum_{k=0}^{n} |u[k]||v[n-k]|$$
$$\leq \sum_{k=0}^{n} C_1 a^k C_2 a^{n-k} = C_1 C_2 (n+1) a^n,$$

and since $\lim_{n\to\infty} n\delta^{-n} = 0$ for any $\delta > 0$, it follows that $|u*v[n]| \leq C b^n$ for every $b > a$.

So the convolution of u and v is defined and belongs to X_b. Taking the Z-transform, we observe that

$$\begin{aligned}\mathcal{Z}(u*v)(z) &= \sum_{n=0}^{\infty} z^{-n} \sum_{k=0}^{n} u[k]v[n-k] = \sum_{n=0}^{\infty}\sum_{k=0}^{n} z^{-k}u[k]z^{-(n-k)}v[n-k] \\ &= \sum_{k=0}^{\infty} z^{-k}u[k] \sum_{n=k}^{\infty} z^{-(n-k)}v[n-k] = /\ n = m+k\ / = \sum_{k=0}^{\infty} z^{-k}u[k]\sum_{m=0}^{\infty} z^{-m}v[m] \\ &= \mathcal{Z}\,v(z)\sum_{k=0}^{\infty} z^{-k}u[k] = \mathcal{Z}\,v(z)\,\mathcal{Z}\,u(z).\end{aligned}$$

□

Example

Show that $\mathcal{Z}\left(\sum_{k=0}^{n} u[k]\right)(z) = \dfrac{z}{z-1}\,\mathcal{Z}\,u(z)$.

Solution. Note that

$$\sum_{k=0}^{n} u[k] = \sum_{k=0}^{n} u[k]\cdot 1 = \sum_{k=0}^{n} u[k]\cdot H[n-k] = u*H[n],$$

so

$$\mathcal{Z}\left(\sum_{k=0}^{n} u[k]\right) = \mathcal{Z}(u*H) = (\mathcal{Z}\,u)(z)\cdot\frac{z}{z-1}.$$

Example

Solve the equation $\sum_{k=0}^{n} u[k]3^{-k} = 6^{-n}$, $n = 0, 1, 2, \ldots$

Solution. Note that we can reformulate the equation as

$$\sum_{k=0}^{n} u[k]3^{-k} = 6^{-n} \quad\Leftrightarrow\quad \sum_{k=0}^{n} u[k]3^{n-k} = 6^{-n}\cdot 3^{n} = 2^{-n}.$$

Taking the Z-transform, we obtain that

$$\mathcal{Z}\,u(z)\,\mathcal{Z}(3^k H[k])(z) = \frac{z}{z-\frac{1}{2}} \quad\Leftrightarrow\quad U(z)\frac{z}{z-3} = \frac{z}{z-\frac{1}{2}}$$

for $|z| > \dfrac{1}{2}$. Hence

$$U(z) = \frac{z-3}{z-\frac{1}{2}} = 1 - \frac{5/2}{z-\frac{1}{2}} = 1 - \frac{5}{2}\,\frac{1}{z}\,\frac{z}{z-\frac{1}{2}}.$$

Therefore it follows that

$$u[k] = \delta[k] - \frac{5}{2}\left(\frac{1}{2}\right)^{k-1} H[k-1],$$

so $u[0] = 1$ and $u[k] = 5\cdot 2^{-k}$ for $k \geq 1$.

12.3 Limit Results

Initial value theorem

Theorem. If there exists some $R > 0$ such that $\mathcal{Z}\, u(z)$ exists for $|z| > R$, then

$$\lim_{|z| \to \infty} \mathcal{Z}\, u(z) = u[0].$$

Proof. Let $|z| > R$ and put $w = z^{-1}$. Then $|w| < R$ and if $|z| \to \infty$ then $w \to 0$. Since $\mathcal{Z}\, u(z)$ converges uniformly for $|z| > R$, it follows that

$$f(w) = \sum_{k=0}^{\infty} u[k] w^k$$

also converges uniformly, so f is a continuous function for $|w| < R$. Hence $\lim_{w \to 0} f(w) = u[0]$ and obviously this limit is the same as $\lim_{|z| \to \infty} \mathcal{Z}\, u(z)$. □

12.4 The Bilateral Z-transform

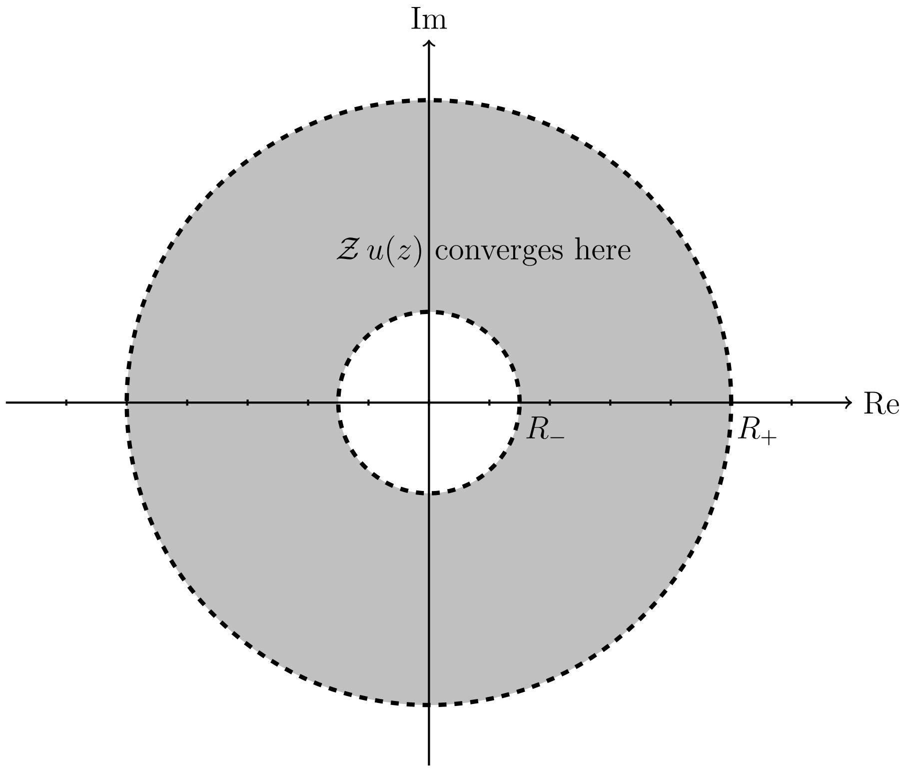
Let $l^1(\mathbf{Z})$ be the space of functions $u \colon \mathbf{Z} \to \mathbf{C}$ such that

$$\sum_{k=-\infty}^{\infty} |u[k]| < \infty,$$

meaning that the sequence is absolutely summable. For functions from this class, we define the **bilateral Z-transform** by

$$\mathcal{Z}\, u(z) = \sum_{k=-\infty}^{\infty} u[k] z^{-k}.$$

The theory is similar to the unilateral transform, but some things change. The region of convergence is not only outside of a disc in this case, but also inside another disc (ideally one that's larger..). Hence the region of convergence looks something like this.



Example

Let $u[k] = a^k$ for $k < 0$ and $u[k] = b^k$ for $k \geq 0$. Find $\mathcal{Z}\,u(z)$. When does the transform exist?

Solution. We find that

$$\begin{aligned}\mathcal{Z}\,u(z) &= \sum_{k=-\infty}^{-1} a^k z^{-k} + \sum_{k=0}^{\infty} b^k z^{-k} = \sum_{k=1}^{\infty} a^{-k} z^k + \frac{1}{1-b/z} = \frac{z}{a}\frac{1}{1-z/a} + \frac{z}{z-b} \\ &= \frac{z}{a-z} + \frac{z}{z-b},\end{aligned}$$

if $|b| < |z| < |a|$.

Inversion works analogously with the unilateral case, making sure that the integration contour is between the two circles.

12.5 The Discrete Time Fourier Transform (DTFT)

By considering $z = e^{i\omega}$, the bilateral Z-transform of $u \in l^1(\mathbf{Z})$ takes the form

$$\mathcal{Z}\,u(e^{i\omega}) = \sum_{k=-\infty}^{\infty} u[k]e^{-ik\omega},$$

which is an absolutely convergent series since $u \in l^1(\mathbf{Z})$. We define the **discrete time Fourier transform (DTFT)** as this expression:

$$\mathcal{F}\,u(\omega) = \sum_{k=-\infty}^{\infty} u[k]e^{-ik\omega}.$$

In a sense, this is the Fourier transform of a function $u\colon \mathbf{Z} \to \mathbf{C}$. Clearly $\mathcal{F}\,u$ is continuous on $\mathbf{R}$, being the uniformly convergent sum of continuous functions, and it is also 2π-periodic:

$$\mathcal{F}\,u(\omega + 2\pi) = \sum_{k=-\infty}^{\infty} u[k]e^{-ik(\omega+2\pi)} = \sum_{k=-\infty}^{\infty} u[k]e^{-ik\omega} = \mathcal{F}\,u(\omega),$$

since this is true for the exponentials in the sum. Moreover, for $u \in l^1(\mathbf{Z})$, the analogous argument with the inversion of the Z-transform shows that

$$u[k] = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}\,u(\omega)e^{i\omega k}\,d\omega, \quad k \in \mathbf{Z}.$$

Prove this!

12.5.1 Connection with Fourier Series

Notice that since $\mathcal{F}\,u(\omega)$ is 2π-periodic and continuous, a natural question would be what the Fourier series looks like. Indeed, the Fourier series of $\mathcal{F}\,u$ is connected with u in the following sense. Suppose that $u \in l^1(\mathbf{Z})$ and let $U(\omega) = \mathcal{F}\,u(\omega)$. Then U has the Fourier series

$$U(\omega) \sim \sum_{k=-\infty}^{\infty} u[-k]e^{ik\omega},$$

and if $u \in E$ is continuous and 2π-periodic with

$$u(x) \sim \sum_{k=-\infty}^{\infty} U[k]e^{ik\omega},$$

then $\mathcal{F}\,U(\omega) = u(-\omega)$ assuming that $U \in l^1(\mathbf{Z})$.

12.6 The Discrete Fourier Transform (DFT)

In the case when a function $u\colon \mathbf{Z} \to \mathbf{C}$ is periodic, meaning that there exists some integer $K > 0$ such that $u[k+K] = u[k]$ for every $k \in \mathbf{Z}$, we can define a variation of the Fourier transform by considering only one period and restricting ourselves to integer values. This variation is usually referred to as the **discrete Fourier transform (DFT)**:

$$\mathcal{F}\,u[n] = \sum_{k=0}^{K-1} u[k]e^{-2\pi ink/K}, \quad n \in \mathbf{Z}.$$

Clearly $\mathcal{F}\,u$ is periodic: $\mathcal{F}\,u[n+K] = \mathcal{F}\,u[n]$, since

$$\begin{aligned}\mathcal{F}\,u[n+K] &= \sum_{k=0}^{K-1} u[k]e^{-2\pi i(n+K)k/K} = \sum_{k=0}^{K-1} u[k]e^{-2\pi ink/K}e^{-2\pi ik} = \sum_{k=0}^{K-1} u[k]e^{-2\pi ink/K}\\ &= \mathcal{F}\,u[n].\end{aligned}$$

Moreover, since both u and $n \mapsto e^{2\pi ink/K}$ (for fixed k) are K-periodic, it follows that

$$\mathcal{F}\,u[n] = \sum_{k=M}^{M+K-1} u[k]e^{-2\pi ink/K}, \quad n \in \mathbf{Z},$$

for any integer M. Note also that the numbers $\omega_k = e^{-2\pi ik/K}$ are the **unit roots**, meaning that for $k = 0, 1, 2, \ldots, K-1$, these numbers are the solutions to the binomial equation $z^K = 1$. The inversion of the discrete Fourier transform is easily carried out by

$$\mathcal{F}^{-1}\, v[n] = \frac{1}{K}\sum_{k=0}^{K-1} v[k]e^{2\pi ink/K}, \quad n \in \mathbf{Z}. \tag{12.1}$$

In the case when we have a function $u\colon \{0, 1, 2, \ldots, K-1\} \to \mathbf{C}$, we proceed like we did when working with Fourier series by considering the periodic extension of u. In this was we can consider the Fourier transform of functions defined on discrete sets. When working with the Fourier transform in applications, this is usually the setting we end up in. Obviously there are a lot of questions as to how the DFT is connected with both the DTFT and the regular Fourier transform on $\mathbf{R}$, but we will not get into these at this point. There are several extremely useful results with regards to sampling of signals that you will see in a course in signal processing.

12.6.1 Circular Convolution

Recall that the previously studied transforms had the nice property that convolutions usually ended up being the product of the transforms of the factors in the convolution. For the DFT, we basically do this "backwards," meaning that we define an operation $\star$ by

$$(u \star v)[n] = \mathcal{F}^{-1}\left(\mathcal{F}\, u\, \mathcal{F}\, v\right)[n].$$

This operation is usually referred to as **circular convolution**. Why circular? This is due to the periodicity of the involved functions u and v when considered as defined on $\mathbf{Z}$. Indeed,

$$(u \star v)[n] = \sum_{k=0}^{K-1} u[k]\, v[(n-k) \mod K].$$

Here $l \mod K = l$ if $0 \le l < K$ and $l \mod K = l - mK$ if there exists an integer m such that $0 \le l - mK < K$.

12.6.2 Properties

Let $U[n] = \mathcal{F}\, u[n]$ and $V[n] = \mathcal{F}\, v[n]$. Then the following properties hold.

(i) Reversal: $U[K-n] = \mathcal{F}(u[K-k])[n]$.

(ii) Conjugation: $\mathcal{F}(\overline{u})[n] = \overline{U[n]}$.

(iii) Parseval's identity:

$$\frac{1}{K}\sum_{k=0}^{K-1} U[k]\,\overline{V[k]} = \sum_{k=0}^{K-1} u[k]\,\overline{v[k]}.$$

(iv) Multiplication: $\mathcal{F}(uv)[n] = \dfrac{1}{K}(\mathcal{F}\, u \star \mathcal{F}\, v)[n]$ (circular convolution).

(v) $\mathcal{F}^2$: $\mathcal{F}(\mathcal{F}\, u)[n] = Ku[n]$.

12.6.3 The Fast Fourier Transform (FFT)

The **fast Fourier transform (FFT)** is not yet another transform, but rather a particular way of calculating the DFT. A naive implementation of the DFT shows that for each value n, calculating $\mathcal{F}\,u[n]$ costs performing a sum of K multiplications. Since there are K unique values for n, the costs of finding the complete DFT would be of order $O(K^2)$ (where the constant does not depend on K). This would make finding the Fourier transform rather expensive if K is large.
The revolutionary (it really was) idea of the FFT is to factor the problem into parts, solving these recursively, and thereby obtaining a complexity of order $O(K \log K)$. This is a huge gain. There are many different algorithms for calculating the DFT and those that has a time complexity of order $O(K \log K)$ are referred to as FFT:s. Let's take a look at one way of handling the case when $K = 2^N$ is a power of 2. If the size is not a perfect power of 2, one can use **zero-padding**, meaning that we extend $u[k]$ by zero until we obtain $K = 2^N$ for some N. How would that affect the DFT?

An example when $K = 2^N$

Since $K = 2^N$, we can split $u[k]$ in two parts: when $k = 2l$ is even and when $k = 2l + 1$ is odd. Note now that

$$\begin{aligned}\mathcal{F}\,u[n] &= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi n(2l)/K} + \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi n(2l+1)/K} \\ &= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi nl/(K/2)} + e^{-i2\pi n/K} \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi nl/(K/2)} \\ &= \mathcal{F}(u[2l])[n] + e^{-i2\pi n/K}\,\mathcal{F}(u[2l+1])[n],\end{aligned}$$

where the last equality assumes that $0 \le n \le K/2 - 1$. For $n \ge K/2$, let $n = m + K/2$ for $m = 0, 1, \ldots, K/2 - 1$. We see that

$$\begin{aligned}\mathcal{F}\,u[m + K/2] &= \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi(m+K/2)(2l)/K} + \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi(m+K/2)(2l+1)/K} \\ &= e^{-i2\pi m} \sum_{l=0}^{K/2-1} u[2l]e^{-i2\pi ml/(K/2)} + e^{-i2\pi(m+K/2)/K} \sum_{l=0}^{K/2-1} u[2l+1]e^{-i2\pi ml/(K/2)} \\ &= \mathcal{F}(u[2l])[m] - e^{-i2\pi m/K}\,\mathcal{F}(u[2l+1])[m].\end{aligned}$$

Hence we have reduced the problem of calculating a DFT of size K to calculating two DFT:s of size $K/2$. This type of recursion will yield a complexity of order $O(K \log K)$.

12.7 Exercises

1. Prove that
$$u \in l^1(\mathbf{Z}) \quad \Rightarrow \quad u[k] = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}\,u(\omega)e^{i\omega k}\,d\omega, \quad k \in \mathbf{Z}.$$

2. Prove that $u[k] = a^{|k|} \in l^1(\mathbf{Z})$ when $|a| < 1$ and find $\mathcal{F}\,u(\omega)$.

3. Prove the inversion formula for the DFT: equation 12.1.

4. Prove the formulas in Section 12.6.2.

Chapter 13

Table of Formulæ

13.1 Notation and Definitions

- $\mathbf{R}$ is the set of all real numbers.
- $\mathbf{Q}$ is the set of all rational numbers.
- $\mathbf{C}$ is the set of all complex numbers.
- $\mathbf{Z}$ is the set of all integers.
- $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers.

For $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$,

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad |z| = \sqrt{x^2 + y^2}.$$

13.1.1 Continuity and Differentiability

- One-sided limits:
$$u(x^{\pm}) = \lim_{x \to x^{\pm}} u(x).$$
- One-sided derivatives:
$$D^{\pm}u(x) = \lim_{h \to 0^{\pm}} \frac{u(x+h) - u(x)}{h}$$
- $C(I)$: The set of all continuous functions on a set I.
- $C^m(I)$: The set of all continuously differentiable (up to order m) functions on a set I.

A function $u \colon I \to \mathbf{C}$ on an interval I is called piecewise continuous if...

- I is finite and there are a finite number of points such that u is continuous everywhere on I except for at these points. Moreover, if $c \in I$ is a point where u is discontinuous, the limits
$$\lim_{I \ni x \to c^-} u(x) \quad \text{and} \quad \lim_{I \ni x \to c^+} u(x)$$
exist (only $u(c^-)$ or $u(c^+)$ if points on the boundary of I).
- I is infinite and there a finite number of exception points (as in the finite case) in each finite sub-interval of I.

13.1.2 Function Spaces

A normed linear space is a linear space V endowed with a norm $\|\cdot\|\colon V \to [0,\infty[$ such that

$$\text{(i) } \|u\| \geq 0 \qquad \text{(ii) } \|\alpha u\| = |\alpha|\|u\|,\ \alpha \in \mathbf{C} \qquad \text{(iii) } \|u+v\| \leq \|u\| + \|v\|.$$

An inner product $\langle\cdot,\ \cdot\rangle : V \times V \to \mathbf{C}$ on a vector space V satisfies

$$\text{(i) } \langle u,\ v\rangle = \overline{\langle v,\ u\rangle} \qquad \text{(ii) } \langle u+v,\ w\rangle = \langle u,\ w\rangle + \langle v,\ w\rangle \qquad \text{(iii) } \langle \alpha u,\ v\rangle = \alpha\langle u,\ v\rangle$$

$$\text{(iv) } \langle u,\ u\rangle \geq 0 \qquad \text{(v) } \langle u,\ u\rangle = 0 \quad \Leftrightarrow \quad u = 0.$$

In an inner product space, we use $\|u\| = \sqrt{\langle u,\ u\rangle}$ as the norm.

Sequence Spaces

The sequence spaces l^p, $1 \leq p \leq \infty$ consists of sequences $(x_1, x_2, x_3, \ldots)$, $x_i \in \mathbf{C}$, such that the norm

$$\|x\|_{l^p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty, 1 \leq p < \infty,$$

or

$$\|x\|_{l^\infty} = \sup_{k\geq 1} |x_k| < \infty.$$

Sometimes $l^p(\mathbf{N})$. The spaces $l^p(\mathbf{Z})$ are defined analogously. Only l^2 is an inner product space with

$$\langle x,\ y\rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$

Lebesgue Spaces (integrable functions)

We define the space $L^1(a,b)$ of absolutely integrable functions $u\colon]a,b[\to \mathbf{C}$ with norm

$$\|f\|_{L^1(a,b)} = \int_a^b |f(x)|\,dx < \infty.$$

The space $L^2(a,b)$ consists of all "square integrable" functions with the norm

$$\|f\|_{L^2(a,b)} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} < \infty.$$

This space is an inner product space with

$$\langle f,\ g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$$

The space $L^\infty(a,b)$ of bounded functions with norm

$$\|f\|_{L^\infty(a,b)} = \sup_{a\leq x\leq b} |f(x)| < \infty.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed and we then write $L^p(\mathbf{R})$. Sometimes we write $\|f\|_p$ instead of $\|f\|_{L^p(a,b)}$.

Spaces of Piecewise Functions

- $E[a,b]$ (or E): The linear space of all piecewise continuous functions on an interval $[a,b]$.
- $E'[a,b]$ (or E'): The linear space of those $u \in E[a,b]$ such that $D^-u(x)$ exists for $a < x \le b$ and that $D^+u(x)$ exists for $a \le x < b$.
- $G(\mathbf{R})$ (or G): The linear space of all piecewise continuous functions on $\mathbf{R}$ that are absolutely integrable on $\mathbf{R}$.

13.1.3 Special Functions

- **Heaviside function**:

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

- **Signum function**:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Discrete Functions

- **Discrete Heaviside function**:

$$H[k] = \begin{cases} 0, & k < 0, \\ 1, & k \ge 0. \end{cases}$$

- **Discrete impulse function**:

$$\delta[k] = \begin{cases} 0, & k \ne 0, \\ 1, & k = 0. \end{cases}$$

- **Binomial coefficient functions**:

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \begin{cases} \frac{n!}{(n-k)!\,k!}, & k = 0, 1, 2, \ldots, \\ 0, & k > n. \end{cases}$$

Convolutions (on R)

The convolution $u * v \colon \mathbf{R} \to \mathbf{C}$ of two functions $u \colon \mathbf{R} \to \mathbf{C}$ and $v \colon \mathbf{R} \to \mathbf{C}$ is defined by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(t) v(x - t)\, dt, \quad x \in \mathbf{R},$$

whenever this integral exists. If $u, v \in G(\mathbf{R})$, then $u * v \in G(\mathbf{R})$.
Suppose that $u, v, w \in G(\mathbf{R})$. Then the convolution has the following properties.

- Associative: $(u * v) * w(x) = u * (v * w)(x)$.
- Distributive: $(u + v) * w(x) = u * w(x) + v * w(x)$.
- Commutative: $u * v(x) = v * u(x)$.

Convolutions (on Z)

- For $u, v\colon \mathbf{Z} \to \mathbf{C}$, the **discrete convolution** $u * v$ is

$$u * v[n] = \sum_{k=-\infty}^{\infty} u[k]v[n-k],$$

whenever this series exists.

- For $u, v\colon \mathbf{N} \to \mathbf{C}$, the **unilateral (or one-sided) discrete convolution** $u * v$ is

$$u * v[n] = \sum_{k=0}^{n} u[k]v[n-k], \quad n = 0, 1, 2, \ldots$$

13.1.4 Inequalities

- **The Cauchy-Schwarz inequality**: If $u, v \in V$ and V is an inner product space, then

$$|\langle u,\ v\rangle| \leq \|u\|\|v\|.$$

- **Bessel's inequality**: Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \ldots\}$ be an ON system in V. Then

$$\sum_{k=1}^{\infty} |\langle v,\ e_k\rangle|^2 \leq \|v\|^2.$$

This implies the **Riemann-Lebesgue lemma** for inner product spaces:

$$\lim_{n\to\infty} \langle v,\ e_n\rangle = 0.$$

- **The triangle inequality**: In a normed space V,

$$|\|u\| - \|v\|| \leq \|u + v\| \leq \|u\| + \|v\|.$$

- **Young's inequality** ($r = p = q = 1$):

$$\|u * v\|_{L^1(\mathbf{R})} \leq \|u\|_{L^1(\mathbf{R})}\|v\|_{L^1(\mathbf{R})}.$$

and

$$\|u * v\|_{l^1(\mathbf{Z})} \leq \|u\|_{l^1(\mathbf{Z})}\|v\|_{l^1(\mathbf{Z})}.$$

13.1.5 Convergence of Sequences

Let $u_1, u_2, \ldots$ be a sequence in a normed space V. We say that $u_n \to u$ for some $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$. This is called **strong convergence** or **convergence in norm**.

Convergence of Functions

- **Pointwise convergence**: We say that $u_k \to u$ pointwise on I as $k \to \infty$ if
$$\lim_{k\to\infty} u_k(x) = u(x)$$
for every $x \in I$. We often refer to u as the *limiting function.*
- **Uniform convergence**: We say that $u_k \to u$ uniformly on $[a, b]$ as $k \to \infty$ if
$$\lim_{k\to\infty} \|u_k - u\|_\infty = 0.$$

Weierstrass' M-test: If $I \subset \mathbf{R}$ and M_k, $k = 1, 2, \ldots$, are constants such that $|u_k(x)| \le M_k$ for $x \in I$, then
$$\sum_{k=1}^{\infty} M_k < \infty \quad \Rightarrow \quad \sum_{k=1}^{\infty} u_k(x) \text{ converges uniformly on } I.$$

If:

- $u_0, u_1, u_2, \ldots$ are continuous functions on $[a, b]$
- and $u(x) = \sum_{k=0}^{\infty} u_k(x)$ is uniformly convergent for $x \in [a, b]$,

then

- the series u is a continuous function on $[a, b]$,
- we can exchange the order of summation and integration:
$$\int_c^d u(x)\,dx = \int_c^d \left(\sum_{k=0}^{\infty} u_k(x)\right) dx = \sum_{k=0}^{\infty} \int_c^d u_k(x)\,dx, \quad \text{for } a \le c < d \le b,$$
- and if in addition $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a, b]$, then
$$u'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} u_k(x)\,dx\right) = \sum_{k=0}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=0}^{\infty} u_k'(x), \quad x \in [a, b].$$

13.1.6 Integration Theory

The **principal value** integral is defined by
$$\text{P.V.} \int_{-\infty}^{\infty} u(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} u(x)\,dx.$$

- If $F(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$ exists for every $x \in I$ and
$$\sup_{x\in I} \left| \int_{-R}^{R} f(x, y)\,dy - F(x) \right| \to 0, \text{ as } R \to \infty,$$
then we call $F(x)$ uniformly convergent on I.

- **Dominated convergence**:

  If:

  - $f\colon \mathbf{R}^2 \to \mathbf{C}$,
  - $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ exists for all x,
  - there is a $g \in L^1(\mathbf{R})$ such that $|f(x,y)| \le g(y)$ for all $x, y \in \mathbf{R}$,

  then $\displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ converges uniformly on $\mathbf{R}$.

- **Continuity**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,R]$. Then

  - $F_R(x) = \displaystyle\int_a^R f(x,y)\,dy$ is continuous on $[c,d]$
  - and if in addition f is continuous on $[c,d] \times [a,\infty[$ and $F(x) = \displaystyle\int_a^{\infty} f(x,y)\,dy$ converges uniformly (on $[c,d]$), then F is continuous.

- **Order of integration**: If $f\colon \mathbf{R}^2 \to \mathbf{C}$ is continuous on $[c,d] \times [a,\infty[$ and $F(x)$ converges uniformly (on $[c,d]$), then

$$\int_c^d \left(\int_a^{\infty} f(x,y)\,dy \right) dx = \int_a^{\infty} \left(\int_c^d f(x,y)\,dx \right) dy.$$

- Note that we can let $a = -\infty$ in the previous theorems by exchanging $[a,R]$ by $[-R,R]$ and consider the principal values.

- **Leibniz's rule**: If

  - $f\colon \mathbf{R}^2 \to \mathbf{C}$ and $f'_x(x,y)$ exist and are continuous,
  - $F(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)\,dy$ is convergent for every x,
  - and $\displaystyle\int_{-\infty}^{\infty} f'_x(x,y)\,dy$ is uniformly convergent,

  then

$$F'(x) = \frac{d}{dx} \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f'_x(x,y)\,dy.$$

13.2 Fourier Series

For $u \in L^1(-\pi, \pi)$:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\sin kx\,dx \quad \text{or} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx$$

are the Fourier coefficients (real or complex) for u. The series

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

is called the **Fourier series** of the function u (real or complex). We write $u(x) \sim S(x)$. Note that:

- if u is even, then $b_k = 0$ for $k = 1, 2, 3, \ldots$;
- if u is odd, then $a_k = 0$ for $k = 1, 2, 3, \ldots$.

If u is a T-periodic function, we define $\Omega = \frac{2\pi}{T}$. The real Fourier series of u is then given by

$$u(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\Omega x + b_k \sin k\Omega x,$$

where

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\cos k\Omega x\,dx \quad \text{and} \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} u(x)\sin k\Omega x\,dx.$$

The complex series is given by

$$u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Omega x}, \quad \text{where } c_k = \frac{1}{T}\int_{-T/2}^{T/2} u(x)e^{-ik\Omega x}\,dx.$$

13.2.1 Parseval's identity

- **Parsevals's identity**:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.

- **Parseval's generalized identity**:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k},$$

  where $u(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ and $v(x) \sim \sum_{k=-\infty}^{\infty} d_k e^{ikx}$.

13.2.2 Convergence

Kernels

- The **Dirichlet kernel**: $D_n(x) = \sum_{k=-n}^{n} e^{ikx},\ x \in \mathbf{R},\ n = 1, 2, 3, \ldots$
- The **Fejér kernel**: $F_n(x) = \frac{1}{n+1} \sum_{l=0}^{n} \sum_{k=-l}^{l} e^{ikx} = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}, \quad n = 0, 1, 2, \ldots$

13.2.3 Convergence Results

- If $u \in L^1(-\pi, \pi)$, then u has a Fourier series.
- Let $u \in E'$. Then

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \to \frac{u(x^+) + u(x^-)}{2}, \quad x \in [-\pi, \pi].$$

- If $u \in E$ and $D^{\pm}u(x)$ exists, then

$$\lim_{n\to\infty} S_n(x) = \frac{u(x^+) + u(x^-)}{2}.$$

- If $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, then $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly.
- If $u \in E$, then $\overline{S_n}(x) \to \frac{u(x^+) + u(x^-)}{2}$.
- If $u, v \in E$ and $\widehat{u}[k] = \widehat{v}[k]$, $k \in \mathbf{Z}$, then $u(x) = v(x)$ whenever u and v are continuous at x.
- If $u' \in E$, u is continuous and $u(-\pi) = u(\pi)$, then $S_n(x)$ converges uniformly to $u(x)$.
- If $u' \in E$ and u is continuous on $[a, b] \subset]-\pi, \pi[$, then $S_n(x)$ converges uniformly on $[a, b]$.
- If $u \in E$ is continuous and $u(-\pi) = u(\pi)$, then $\overline{S_n}(x)$ converges uniformly to $u(x)$.

13.2.4 General Fourier Series

- For a given ON system, the complex numbers $\langle v,\ e_k\rangle$, $k = 1, 2, \ldots$, are called the **generalized Fourier coefficients** of v.
- If $W = \{e_1, e_2, \ldots\}$ is an ON system in V, then W is closed if and only if **Parseval's identity** holds:

$$\sum_{k=1}^{\infty} |\langle v,\ e_k\rangle|^2 = \|v\|^2, \quad v \in V,$$

  or if $a_k = \langle u,\ e_k\rangle$ and $b_k = \langle v,\ e_k\rangle$, then

$$\langle u,\ v\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

13.2.5 Rules for Fourier Coefficients

Let $u, v \in E$ be periodic with period $T > 0$ and define $\Omega = 2\pi/T$.

Table 13.1: Rules for Fourier Coefficients

Function	Fourier coefficient	Notes
$c_1 u(x) + c_2 v(x)$	$c_1 U[k] + c_2 V[k]$	
$(u * v)(x)$	$U[k]V[k]$	
$u(x)v(x)$	$(U * V)[n]$	
$e^{im\Omega x} u(x)$	$U[k-m]$	$m \in \mathbf{Z}$
$u(ax)$ (per. T/a)	$U[k]$	
$u(-x)$	$F[-k]$	
$\overline{u(x)}$	$\overline{U[-k]}$	
$u'(x)$	$ik\Omega U[k]$	
$u^{(n)}(x)$	$(i\Omega k)^n U[k]$	$n = 1, 2, \ldots$

13.3 The Fourier Transform

The **Fourier transform** of a function $u\colon \mathbf{R} \to \mathbf{C}$ given by

$$U(\omega) = \widehat{u}(\omega) = \mathcal{F}\, u(\omega) = \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\, dx, \quad \omega \in \mathbf{R},$$

when this integral exists.

- If $u \in L^1(\mathbf{R})$ then $\mathcal{F}\, u(\omega)$ exists for all $\omega \in \mathbf{R}$ and

$$\| \mathcal{F}\, u\|_\infty \le \|u\|_{L^1(\mathbf{R})}.$$

- For $u \in G$, the Fourier transform $\mathcal{F}\, u$ is uniformly continuous on $\mathbf{R}$.
- **The Riemann-Lebesgue lemma**: For $u \in G$ we have $\mathcal{F}\, u(\omega) \to 0$ as $|w| \to \infty$.

13.3.1 Convergence

Kernels

- The **Dirichlet kernel** (on $\mathbf{R}$):

$$D_R(x) = \frac{\sin(Rx)}{\pi x}, \quad x \neq 0,$$

  and $D_R(0) = R/\pi$.
- The **Fejér kernel (on R)**:

$$F_M(t) = \frac{1}{2\pi}\int_{-M}^{M} \left(1 - \frac{|\omega|}{M}\right) e^{i\omega t}\, d\omega = \frac{1-\cos Mx}{\pi M x^2} = \frac{M}{2\pi}\left(\frac{\sin(Mx/2)}{Mx/2}\right)^2,$$

  the last two equalities assumes that $x \neq 0$.

Inversion

- If $u \in G(\mathbf{R})$ and $D^{\pm}u(x)$ exists, then

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \mathcal{F}\, u(\omega)e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \mathcal{F}\, u(\omega)\left(1 - \frac{|\omega|}{R}\right) e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2}.$$

- If $u \in G(\mathbf{R})$, then

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \mathcal{F}\, u(\omega)e^{i\omega x}\, d\omega = \frac{u(x^+) + u(x^-)}{2},$$

  whenever the limit exists.
- **Uniqueness**: If $u, v \in G(\mathbf{R})$ and $\mathcal{F}\, u(\omega) = \mathcal{F}\, v(\omega)$, $\omega \in \mathbf{R}$, then $u(x) = v(x)$ for every $x \in \mathbf{R}$ where both u and v are continuous.

13.3.2 Special Rules

- If $u, U \in G(\mathbf{R})$ and $U(\omega) = \mathcal{F}(u)(\omega)$, then

$$\mathcal{F}^{-1}(U)(x) = \frac{1}{2\pi}\,\mathcal{F}\big((\mathcal{F}\,u)(-\omega)\big)(x) \quad \text{and} \quad \mathcal{F}(\mathcal{F}\,u(\omega))(x) = 2\pi u(-x),$$

  for every x where u is continuous and $D^{\pm}u(x)$ exist.

- If $u, v \in G(\mathbf{R})$ such that $uv, \mathcal{F}\,u, \mathcal{F}\,v \in G(\mathbf{R})$, then

$$\mathcal{F}(uv)(\omega) = \frac{1}{2\pi}\,\mathcal{F}(u) * \mathcal{F}(v)(\omega).$$

13.3.3 Plancherel's formula

- If $u \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then $\mathcal{F}\,u \in L^2(\mathbf{R})$.
- **Plancherel's formula**: If $u, v \in G(\mathbf{R}) \cap L^2(\mathbf{R})$, then

$$\int_{-\infty}^{\infty} u(x)\,\overline{v(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\,u(\omega)\,\overline{\mathcal{F}\,v(\omega)}\,d\omega.$$

13.3.4 Rules for the Fourier Transform

Let $U(\omega) = \mathcal{F}\,u(\omega)$ and $V(\omega) = \mathcal{F}\,v(\omega)$.

Table 13.2: Rules for Fourier transform

Function	Fourier transform	Notes
$c_1u(x) + c_2v(x)$	$c_1U(\omega) + c_2V(\omega)$	
$(u * v)(x)$	$U(\omega)V(\omega)$	
$e^{iax}u(x)$	$U(\omega - a)$	$a \in \mathbf{R}$
$u(x)\cos ax$	$\dfrac{U(\omega + a) + U(\omega - a)}{2}$	$a \in \mathbf{R}$
$u(x)\sin ax$	$\dfrac{U(\omega + a) - U(\omega - a)}{2i}$	$a \in \mathbf{R}$
$u(x - x_0)$	$e^{-ix_0\omega}U(\omega)$	$x_0 \in \mathbf{R}$
$u(ax)$	$\dfrac{1}{\lvert a\rvert}\,U\left(\dfrac{\omega}{a}\right)$	$a \in \mathbf{R}$, $a \neq 0$
$\overline{u(x)}$	$\overline{U(\overline{\omega})}$	
$u'(x)$	$i\omega U(\omega)$	$u \in C(\mathbf{R})$, $u' \in G$
$u^{(n)}(x)$	$(i\omega)^n U(\omega)$	...
$x^m u(x)$	$i^m U^{(m)}(\omega)$	$x^m u(x) \in G$, $m = 1, 2, 3, \dots$

13.3.5 Fourier Transforms

Table 13.3: Fourier transforms

Function	Fourier transform	Notes
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$	$a > 0$
$e^{-a\|x\|}$	$\frac{2a}{a^2+\omega^2}$	$a > 0$
$\operatorname{sgn}(x)e^{-a\|x\|}$	$\frac{-2i\omega}{a^2+\omega^2}$	$a > 0$
$H(x)e^{-ax}$	$\frac{1}{a+i\omega}$	$\operatorname{Re} a > 0$
$H(-x)e^{ax}$	$\frac{1}{a-i\omega}$	$\operatorname{Re} a > 0$
$\frac{1}{a^2+x^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$	$a > 0$
$H(x+a) - H(x-a)$	$\frac{2\sin a\omega}{\omega}$	$a > 0$
$\operatorname{sgn}(x)\big(H(x+a) - H(x-a)\big)$	$\frac{2(1-\cos a\omega)}{i\omega}$	$a > 0$
$(a - \|x\|)\big(H(x+a) - H(x-a)\big)$	$\frac{2(1-\cos a\omega)}{\omega^2}$	$a > 0$
$\frac{1-\cos at}{t^2}$	$\pi(a - \|\omega\|)\big(H(\omega+a) - H(\omega-a)\big)$	$a > 0$

13.4 The (unilateral) Laplace Transform

The Laplace transform of $u\colon [0,\infty[\to \mathbf{C}$ is given by

$$\mathcal{L}\, u(s) = \int_0^\infty u(t)e^{-st}\, dt,$$

for those $s = \sigma + i\omega \in \mathbf{C}$, $\sigma, \omega \in \mathbf{R}$, where this integral is convergent.

- **Exponential growth**: A piecewise continuous $u\colon [0,\infty[$ is of exponential growth (of order a) if there exists constants $a > 0$ and $K > 0$ such that $|u(t)| \le Ke^{at}$ for $t \ge 0$. The set of all such functions will be denoted by X_a.
- **Existence of $\mathcal{L}\, u(s)$**: If $u \in X_a$ for some $a > 0$, then the Laplace transform $\mathcal{L}\, u(s)$ exists (at least) for $\operatorname{Re} s > a$.
- $\mathcal{L}\, u(s) \to 0$ as $\mathbf{R} \ni s \to \infty$.
- $\mathcal{L}\, u(s)$ converges uniformly for $\operatorname{Re} s > a$.
- $\mathcal{L}\, u(s)$ is analytic for $\operatorname{Re} s > a$.
- **Periodicity**. If there exists $T > 0$ such that $u(t + T) = u(t)$ for every $t \ge 0$, then

$$\mathcal{L}\, u(s) = \frac{1}{1 - e^{-sT}} \int_0^T u(\tau)e^{-s\tau}\, d\tau.$$

13.4.1 Inversion

- If $u \in X_a$ has right- and lefthand limits at a point $t > 0$, then

$$\lim_{L\to\infty} \frac{1}{2\pi} \int_{-L}^{L} \mathcal{L}\, u(\sigma + i\omega)e^{\sigma t}e^{i\omega t}\, d\omega = \frac{u(t^+) + u(t^-)}{2},$$

  where the vertical line $\operatorname{Re} z = \sigma$ is contained in the region of convergence of $\mathcal{L}\, u(s)$
- If $u, v \in X_a$ and $\mathcal{L}\, u(s) = \mathcal{L}\, v(s)$ on some vertical line $\operatorname{Re} s = \sigma$, then $u(t) = v(t)$ for all t where u and v are continuous.

13.4.2 Limit Theorems

- **Final value theorem**:
  If $u\colon [0,\infty[\to \mathbf{C}$ is bounded and $\lim_{t\to\infty} u(t) = A$, then $A = \lim_{\mathbf{R}\ni s\to 0^+} s\, \mathcal{L}\, u(s)$.
- **Initial value theorem**:
  If $u\colon [0,\infty[\to \mathbf{C}$ belongs to X_b and $\lim_{t\to 0^+} u(t) = a$, then $a = \lim_{\mathbf{R}\ni s\to\infty} s\, \mathcal{L}\, u(s)$.

13.4.3 Rules for the Laplace Transform

Let $U(s) = \mathcal{L}\, u(t)$, $\sigma > \sigma_u$ and $V(s) = \mathcal{L}\, v(t)$, $\sigma > \sigma_v$.

Table 13.4: Rules for Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$c_1 u(t) + c_2 v(t)$	$c_1 U(s) + c_2 V(s)$	$\sigma > \max\{\sigma_u, \sigma_v\}$
$(u * v)(t)$	$U(s)V(s)$	$\sigma > \max\{\sigma_u, \sigma_v\}$
$e^{at}u(t)$	$U(s-a)$	$\sigma > \sigma_u + \operatorname{Re} a$
$u(t-t_0)H(t-t_0)$	$e^{-t_0 s}U(s)$	$\sigma > \sigma_u$ $a > 0$
$u(at)$	$\frac{1}{a} U\left(\frac{s}{a}\right)$	$\sigma > a\sigma_u$, $a > 0$
$\overline{u(t)}$	$\overline{U(\overline{s})}$	$\sigma > \sigma_u$
$u'(t)$	$sU(s) - u(0)$	$\sigma > \sigma_u$
$u^{(n)}(t)$	$s^n U(s) - s^{(n-1)}u(0) - \cdots$ $-su^{(n-2)}(0) - u^{(n-1)}(0)$	$\sigma > \max\{\sigma_u, \sigma_{u'}, \ldots, \sigma_{u^{(n-1)}}\}$
$\int_0^t u(\tau)\, d\tau$	$\frac{U(s)}{s}$	$\sigma > \max\{\sigma_u, 0\}$
$t^m u(t)$	$(-1)^m U^{(m)}(s)$	$\sigma > \sigma_u$

13.4.4 Laplace Transforms

Table 13.5: Laplace transforms

Function	Unilateral Laplace transform	Region of convergence
$H(t) = 1$	$\frac{1}{s}$	$\sigma > 0$
t	$\frac{1}{s^2}$	$\sigma > 0$
t^m	$\frac{m!}{s^{m+1}}$	$\sigma > 0$
		$m = 1, 2, 3, \ldots$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\sigma > 0$
		$a > 0$
e^{at}	$\frac{1}{s-a}$	$\sigma > \operatorname{Re} a$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}$	$\sigma > \operatorname{Re} a$
		$m = 1, 2, 3, \ldots$
$\cos at$	$\frac{s}{s^2+a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\frac{\sin at}{t}$	$\arctan\left(\frac{a}{s}\right)$	$\sigma > \lvert \operatorname{Im} a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$\sinh at$	$\frac{1}{s^2-a^2}$	$\sigma > \lvert \operatorname{Re} a \rvert$
$J_0(at)$	$\frac{1}{\sqrt{1+s^2}}$	$\sigma > \lvert \operatorname{Im} a \rvert$

13.5 The (unilateral) Z Transform

The **Z transform** of a sequence $u[k]$, $k = 0, 1, 2, \ldots$, is defined by

$$\mathcal{Z}(u)(z) = \sum_{k=0}^{\infty} u[k]\, z^{-k},$$

for those $z = x + iy \in \mathbf{C}$, $x, y \in \mathbf{R}$, where this series is absolutely convergent.

- **Existence of $\mathcal{Z}\,u(z)$**: For a sequence $u[k]$, $k = 0, 1, 2, \ldots$, the Z transform $\mathcal{Z}\,u(z)$ has a region of convergence R such that $\mathcal{Z}\,u(z)$ is absolutely (uniformly) convergent for $|z| > R$ and divergent for $|z| < R$. It is possible that $R = 0$ or $R = \infty$.
- **Inversion**: If $U(z) = \mathcal{Z}\,u(z)$, then

$$u[k] = \frac{1}{2\pi i} \oint_{\gamma} z^{k-1} U(z)\, dz, \quad k = 0, 1, 2, \ldots$$

- **Uniqueness**: If $\mathcal{Z}\,u(z) = \mathcal{Z}\,v(z)$ for all $|z| > R$ for some $R > 0$, then $u[k] = v[k]$ for $k = 0, 1, 2, \ldots$.
- **Initial value theorem**: If there's an $R > 0$ such that $\mathcal{Z}\,u(z)$ exists for $|z| > R$, then

$$\lim_{|z|\to\infty} \mathcal{Z}\,u(z) = u[0].$$

13.5.1 Rules for the Z Transform

Let $U(z) = \mathcal{Z}(u[k])(z)$, $|z| > R_u$ and $V(z) = \mathcal{Z}(v[k])(z)$, $|z| > R_v$.

Table 13.6: Rules for Z transforms

Function	Unilateral Z transform transform	Region of convergence
$c_1 u[k] + c_2 v[k]$	$c_1 U(z) + c_2 V(z)$	$\lvert z\rvert > \max\{R_u, R_v\}$
$(u * v)[k]$	$U(z)V(z)$	$\lvert z\rvert > \max\{R_u, R_v\}$
$a^k u[k]$	$U\left(\frac{z}{a}\right)$	$\lvert z\rvert > \lvert a\rvert R_u$, $a \neq 0$
$u[k-m]H[k-m]$	$z^{-m}U(z)$	$\lvert z\rvert > R_u$, $m = 1, 2, 3, \ldots$
$u[k-m]$	$z^{-m}U(z) + z^{-m+1}u[-1] + \cdots$ $\cdots + z^{-1}u[-m+1] + u[-m]$	$\lvert z\rvert > R_u$, $m = 1, 2, 3, \ldots$
$u[k+m]$	$z^m U(z) - z^m u[0] + \cdots$ $\cdots - z^2 u[m-2] - z u[m-1]$	$\lvert z\rvert > R_u$, $m = 1, 2, 3, \ldots$
$\overline{u[k]}$	$\overline{U(\overline{z})}$	$\lvert z\rvert > R_u$
$\sum_{l=0}^{k} u[l]$	$\frac{z}{z-1} U(z)$	$\lvert z\rvert > \max\{R_u, 1\}$
$k^m u[k]$	$\left(-z\frac{d}{dz}\right)^m U(z)$	$\lvert z\rvert > R_u$

13.5.2 Z Transforms

Table 13.7: Z transforms

Function	Unilateral Z transform	Region of convergence
$\delta[k]$	1	$z \in \mathbf{C}$
$\delta[k-m]$	z^{-m}	$\|z\| > 0$, $m = 1, 2, \dots$
$H[k]$	$\dfrac{z}{z-1}$	$\|z\| > 1$
k	$\dfrac{z}{(z-1)^2}$	$\|z\| > 1$
a^k	$\dfrac{z}{z-a}$	$\|z\| > \|a\|$
ka^k	$\dfrac{az}{(z-a)^2}$	$\|z\| > \|a\|$
k^2a^k	$\dfrac{az^2 + a^2z}{(z-a)^3}$	$\|z\| > \|a\|$
k^3a^k	$\dfrac{az^3 + 4a^2z^2 + a^3z}{(z-a)^4}$	$\|z\| > \|a\|$
$(k+1)a^k$	$\dfrac{z^2}{(z-a)^2}$	$\|z\| > \|a\|$
$\binom{k+m}{m} a^k$	$\dfrac{z^{m+1}}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \dots$
$\binom{k}{m} a^k$	$\dfrac{a^m z}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \dots$
$\binom{k+n}{m} a^k$	$\dfrac{a^{m-n}z^{n+1}}{(z-a)^{m+1}}$	$\|z\| > \|a\|$, $m = 2, 3, \dots,$ $n = 1, \dots, m-1$
$\cos k\alpha$	$\dfrac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$	$\|z\| > 1$
$\sin k\alpha$	$\dfrac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$	$\|z\| > 1$
$k\cos k\alpha$	$\dfrac{z^3\cos\alpha - 2z^2 + z\cos\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\|z\| > 1$
$k\sin k\alpha$	$\dfrac{z^3\sin\alpha - z\sin\alpha}{(z^2 - 2z\cos\alpha + 1)^2}$	$\|z\| > 1$
$\dfrac{a^k}{k!}$	$e^{a/z}$	$\|z\| > 0$
$\dfrac{1}{k}H[k-1]$	$\ln\dfrac{z}{z-1}$	$\|z\| > 1$
$\binom{n}{k} a^k b^{n-k}$	$\dfrac{(bz+a)^n}{z^n}$	$\|z\| > 0$, $n = 1, 2, 3, \dots$

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